The Katowice Problem

Quidquid latine dictum sit, altum videtur

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Exercise 1

Let X and Y be sets and assume $b : \mathcal{P}(X) \to \mathcal{P}(Y)$ is a bijection.

Construct a bijection $c : X \to Y$

Exercise 2

Let X and Y be sets and assume $b : \mathcal{P}(X) \to \mathcal{P}(Y)$ is a bijection that preserves $\subseteq$ (both ways).

Construct a bijection $c : X \to Y$

Note that b is actually an isomorphism of Boolean algebras.

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Exercise 3

Let X and Y be sets and assume $b : \mathcal{P}(X)/\mathit{fin} \to \mathcal{P}(Y)/\mathit{fin}$ is an isomorphism of Boolean algebras.

Construct a bijection $c : X \to Y$

Note: *fin* is the ideal of finite subsets of X (or Y).

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Solution 1

There is no such construction, *because* it is consistent to have X and Y and a bijection $b : \mathcal{P}(X) \to \mathcal{P}(Y)$, but no bijection $c : X \to Y$.

Remember what Mirna said when she quoted Ken Kunen.

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Solution 2

No construction needed.

The bijection c is hiding in plain sight:
the points of X are the atoms of $\mathcal{P}(X)$ and b maps these to the atoms of $\mathcal{P}(Y)$, that is, to the points of Y.

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Solution 3

I know of no construction, even though we know that, in the majority of cases, a bijection c must exist.

Start at the bottom

Theorem

If $\lambda > \omega$ and $\mathcal{P}(\omega)/fin$ and $\mathcal{P}(\lambda)/fin$ are isomorphic then $\lambda = \omega_1$.

Proof.

If $\lambda \geqslant \omega_2$ then $\mathcal{P}(\omega)/fin$, $\mathcal{P}(\omega_1)/fin$ and $\mathcal{P}(\omega_2)/fin$ are all isomorphic.
Reason: $\mathcal{P}(\omega)/fin$ is isomorphic with $\mathcal{P}(A)/fin$ for every infinite subset A of ω. □

Continued

From now on we write A^* for $\mathcal{P}(A)/\mathit{fin}$.

Proof, part 2.

From the state isomorphism that we have for ω^*, ω_1^* and ω_2^*, we deduce that $\mathfrak{d} = \aleph_1$ and $\mathfrak{d} = \aleph_2$, in particular: $\aleph_1 = \aleph_2$, which you'll all agree flies in the face of common sense. □

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Higher up

Theorem

If κ and λ are infinite cardinals such that $\kappa \leqslant \lambda$ and such that κ^* and λ^* are isomorphic then $\lambda \leqslant \max\{\kappa, \aleph_1\}$.

Proof.

By induction on κ.
True for $\kappa = \aleph_0$.
Assume $\kappa \geqslant \aleph_1$ and true for all $\mu < \kappa$.
Let $\lambda > \kappa$ and assume $h : \kappa^* \to \lambda^*$ is an isomorphism.
For $\alpha < \kappa$ choose $A_\alpha \subseteq \lambda$ such that $b(\alpha^*) = A_\alpha^*$.
By assumption $|A_\alpha| \leqslant \max\{|\alpha|, \aleph_1\}$ □

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Higher up

Proof, continued.

Let $A = \bigcup_\alpha A_\alpha$ and $B = \lambda \setminus A$, also choose $C \subseteq \kappa$ with $h(C^*) = B^*$.
Then $|A| \leqslant \kappa$ and $|B| = \lambda$.
Also, as $B \cap A_\alpha = \emptyset$ we have $|C \cap \alpha| < \aleph_0$ for all α.
But this gives us $|C| \leqslant \aleph_0$, and so, by what we had already established: $|B| \leqslant \aleph_1$.
Contradiction. $\square$

Summary

If $\omega_1 \leqslant \kappa < \lambda$ then κ^* and λ^* are not isomorphic, and if $\omega_2 \leqslant \lambda$ then ω^* and λ^* are not homeomorphic.

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A structure on ω

Work with the set $D = \mathbb{Z} \times \omega_1$ — so we have an isomorphism $h : D^* \to \omega^*$.

On the D-side we have

- $V_n = \{n\} \times \omega_1$, for $n \in \mathbb{Z}$
- $H_\alpha = \mathbb{Z} \times \{\alpha\}$, for $\alpha \in \omega_1$
- $B_\alpha = \mathbb{Z} \times \alpha$, for $\alpha \in \omega_1$
- $E_\alpha = \mathbb{Z} \times [\alpha, \omega_1)$, for $\alpha \in \omega_1$

and an automorphism σ of D^* induced by the map $\langle n, \alpha \rangle \mapsto \langle n+1, \alpha \rangle$.

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A structure on ω

On the ω-side we get

- v_n such that $v_n^* = h(V_n^*)$, for $n \in \mathbb{Z}$
- h_α such that $h_\alpha^* = h(H_\alpha^*)$, for $\alpha \in \omega_1$
- b_α such that $b_\alpha^* = h(B_\alpha^*)$, for $\alpha \in \omega_1$
- e_α such that $e_\alpha^* = h(E_\alpha^*)$, for $\alpha \in \omega_1$

and the automorphism τ of ω^* given by $\tau = h \circ \sigma \circ h^{-1}$

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Properties of the structure

We can replace ω by $\mathbb{Z} \times \omega$, so that $v_n = \{n\} \times \omega$.

$\{h_\alpha : \alpha < \omega_1\}$ is an almost disjoint family such that whenever we have $x_\alpha \subseteq h_\alpha$ for all α then there is an X such that $x \cap h_\alpha =^* x_\alpha$ for all α.

This is a very strong property and implies, among others, that $2^{\aleph_0} = 2^{\aleph_1}$.

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Scales

For $\alpha < \omega_1$ define $f_\alpha : \mathbb{Z} \to \omega$ by

$$f_\alpha(n) = \min\{m : \langle n, m \rangle \in e_\alpha\}$$

then $\langle f_\alpha : \alpha < \omega_1 \rangle$ is an ω_1-scale: increasing and cofinal with respect to the order $\leqslant^*$.

$f \leqslant^* g$ means $\{n : f(n) > g(n)\}$ is finite.

In terms of small cardinals: $\mathfrak{d} = \aleph_1$.

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The automorphism τ

First: $\tau(v_n^*) = v_{n+1}^*$ for all n

The equivalence classes h_α^* form a maximal disjoint family of τ-invariant elements

τ is not trivial: there is no bijection between cofinite subsets of ω that induces τ.

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To work

It seems to me that all these things should not be able to coexist in this one structure.

Homework for next year: prove me wrong, or, preferably, prove me right and thus show that $\mathcal{P}(\omega)/\mathit{fin}$ and $\mathcal{P}(\omega_1)/\mathit{fin}$ are not isomorphic.