A class of invisible spaces

Quidquid latine dictum sit, altum videtur

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Small diagonals

Definition (Van Douwen, Hušek, Zhou)

A space, X, has a *small diagonal* if every uncountable subset of $X^2 \setminus \Delta(X)$ has an uncountable subset whose closure is disjoint from $\Delta(X)$.

Hušek defined the negation: X has an ω_1-accessible diagonal is there if a sequence $\bigl\langle \langle x_\alpha, y_\alpha \rangle : \alpha \in \omega_1 \bigr\rangle$ that converges to $\Delta(X)$

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A sufficient condition

If $\Delta(X)$ is a G_δ-set then X has a small diagonal.
Hence, for example, metrizable spaces have small diagonals.

Raison d'etre

Theorem (Hušek, special case)

If $f : \prod_{i \in I} X_i \to X$ is continuous, X and all X_i are compact and X has a small diagonal then f depends on countably many coordinates.

Here is a heavy-handed proof.
Assume not ... Contradiction.

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Details

Take a sequence $\langle M_\alpha : \alpha \in \omega_1 \rangle$ of countable elementary substructures of a suitable $H(\theta)$ such that
$\langle X_i : i \in I \rangle$, f and X belong to M_0,
and also $\langle M_\beta : \beta \leqslant \alpha \rangle \in M_{\alpha+1}$ (all α)
and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ (limit α).

Apply "Assume not" at each α: there are x_α and y_α that have the same coordinates in $I \cap M_\alpha$ but satisfy $f(x_\alpha) \neq f(y_\alpha)$.

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Details (continued)

Take basic neighbourhoods V_α and W_α of x_α and y_α such that $f[V_\alpha] \cap f[W_\alpha] = \emptyset$.

Now redefine y_α so that it agrees with x_α outside the union, F_α, of the supports of V_α and W_α.

Then still $f(x_\alpha) \neq f(y_\alpha)$ but now the points differ in finitely many places.

By elementarity x_α, y_α, V_α and W_α may be taken in $M_{\alpha+1}$, and so $F_\alpha \subseteq M_{\alpha+1} \setminus M_\alpha$.

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Details (finished)

Take an uncountable subset A of ω_1 such that $\operatorname{cl}\bigl\{\bigl\langle f(x_\alpha), f(y_\alpha)\bigr\rangle : \alpha \in A\bigr\}$ is disjoint from $\Delta(X)$.

Take $x \in \prod_{i\in I} X_i$ such that $\{\alpha \in A : x_\alpha \in U\}$ is uncountable, for every neighbourhood U of x.

Take a basic neighbourhood U of x such that $f[U]^2$ is disjoint from the closure above.

Take $\alpha \in A$ such that F_α is disjoint from the support of U, yet $x_\alpha \in U$, then also $y_\alpha \in U$, and $\bigl\langle f(x_\alpha), f(y_\alpha)\bigr\rangle \in f[U]^2$.

Contradiction!

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Elementary sequences

Many proofs of results on csD spaces (compact small Diagonal) work like this.

Let X be compact; a sequence $\langle M_\alpha : \alpha \in \omega_1 \rangle$ of countable elementary substructures, as above, with $X \in M_0$ is an *elementary sequence for* X.

We assume X to be embedded into $[0,1]^\kappa$, say $\kappa = w(X)$, so the following makes sense.

An *elementary sequence of pairs for X* is a sequence $\bigl\langle \{x_\alpha, y_\alpha\} : \alpha \in \omega_1 \bigr\rangle$ such that, always,

$x_\alpha \restriction M_\alpha = y_\alpha \restriction M_\alpha$,

$x_\alpha \neq y_\alpha$, and

$\{x_\alpha, y_\alpha\} \in M_{\alpha+1}$.

One of the coordinate sequences may be constant.

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Elementary sequences and small Diagonals

Gruenhage: a compact space is csD iff every ω_1-sequence of pairs *is ω_1-separated*, i.e., there is an uncountable set A such that $\mathrm{cl}\{x_\alpha : \alpha \in A\}$ and $\mathrm{cl}\{y_\alpha : \alpha \in A\}$ are disjoint.

In fact: a compact space is csD iff every elementary sequence of pairs is ω_1-separated.

In case the x_α are the same this means that $\langle y_\alpha : \alpha \in \omega_1 \rangle$ will have many complete accumulation points.

Now go back to the proof and recognize all these ingredients.

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Remember: a compact space is metrizable iff its diagonal is a G_δ-set.

Also: a compact space is metrizable iff every elementary sequence of pairs does not exist.

We are lead to ask

Is every csD space metrizable?

I will discuss various positive consistency results.

There is as yet no consistent *negative* answer.

This explains the title of this talk: there are no illuminating examples of csD spaces; for all we know they are all metrizable.

Positive answers

If X is csD and $w(X) \leqslant \aleph_1$ then X is metrizable, because X is not csD if $w(X) = \aleph_1$.

We assume $X \subseteq [0,1]^{\omega_1}$, we take an elementary sequence, and with it an elementary sequence of pairs.

Note: $\omega_1 \subseteq \bigcup_\alpha M_\alpha$.

Let $A \subseteq \omega_1$ be uncountable and let x be such that $\{\alpha \in A : x_\alpha \in V\}$ is uncountable, for every basic neighbourhood V of x.

Let V be such a neighbourhood and pick δ such that V is supported in M_δ.

Then $x_\alpha \in V$ iff $y_\alpha \in V$ for $\alpha \geqslant \delta$, hence $x \in \mathrm{cl}\{x_\alpha : \alpha \in A\} \cap \mathrm{cl}\{y_\alpha : \alpha \in A\}$.

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Continuum Hypothesis

Hušek: if X is csD then X is metrizable in each of the following cases

- X has countable tightness
- X is separable

Separable: $w(X) \leqslant 2^{\aleph_0} = \aleph_1$, so there
Countable tightness: it implies separability in this case

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Continuum Hypothesis

Juhász and Szentmiklóssy: if X has uncountable tightness then there is a free ω_1-sequence $\langle x_\alpha : \alpha \in \omega_1 \rangle$ that converges, to x say.

Then $\bigl\langle \{x, x_\alpha\} : \alpha \in \omega_1 \bigr\rangle$ would not be ω_1-separated.

Hence: csD spaces have countable tightness and the Continuum Hypothesis implies csD spaces are metrizable.

Proper Forcing Axiom

Dow and Pavlov: PFA implies csD spaces are metrizable.

I'm not even attempting to sketch the proof.

A sufficient condition

Gruenhage: If X is csD and hereditarily Lindelöf (equivalently, perfectly normal) then X is metrizable.

Take an elementary sequence of pairs and $A \subseteq \omega_1$ uncountable.
Then $\{x_\alpha : \alpha \in A\}$ is Lindelöf.
Pick $\delta \in A$ such that $\{\alpha \in A : x_\alpha \in V\}$ is uncountable, for every basic neighbourhood V of x_δ.
$M_{\delta+1}$ contains a countable local base, $\mathcal{B}$, at x_δ.
For each member B of $\mathcal{B}$ and $\alpha > \delta$ we have $x_\alpha \in B$ iff $y_\alpha \in B$.

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A sufficient condition

Let us squeeze whatever we can out of that proof, put $M = \bigcup_\alpha M_\alpha$.

It suffices that X be first-countable and $X \cap M$ be Lindelöf: we don't need x_δ, just an x in $X \cap M$.

For that Lindelöfness of $X \cap M$ is enough.

We do need a countable local base at x to make the last part work.

A sufficient condition

It suffices that $X \cap M$ be Lindelöf for just one elementary sequence. For then we can prove that our csD space X is first-countable.

If X is not then some $x \in X \cap M_0$ does not have a countable local base.

Hence we can choose $x_\alpha \in M_{\alpha+1}$ such that $x_\alpha \neq x$, but $x_\alpha \restriction M_\alpha = x \restriction M_\alpha$.

Let $A \subseteq \omega_1$ be uncountable and let $y \in M$ be a complete accumulation point of $\{x_\alpha : \alpha \in A\}$.

A sufficient condition

It follows that $y \restriction M_\alpha = x \restriction M_\alpha$ for all α and hence $x \restriction M = y \restriction M$.

By elementarity: $x = y$.

The sequence $\langle x_\alpha : \alpha \in \omega_1 \rangle$ converges to x.

Contradiction.

Hušek asked: does every compact Hausdorff space have either a convergent ω-sequence or a convergent ω_1-sequence.

Call a space ω_1-free of it contains no convergent ω_1-sequences.

In particular csD spaces are ω_1-free.

A consistent “Yes” to Hušek

In any extension of a model of the Continuum Hypothesis by a property K forcing every ω_1-free compact space is L-reflecting (there is an elementary sequence for it such that $X \cap M$ is Lindelöf) and (hence) first-countable.
In particular csD spaces are metrizable in these extensions.

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However

The main question remains: are compact csD spaces metrizable.

(Many partial questions do not have answers yet either: somewhere/everywhere first-countable, what is the weight, ...)

I will be expecting solutions from you next year.

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Light reading

Website: `fa.its.tudelft.nl/~hart`

Alan Dow and Klaas Pieter Hart,
Elementary chains and compact spaces with a small diagonal, Indagationes Mathematicae, **23** (2012), 438–447.

Alan Dow and Klaas Pieter Hart,
Reflecting Lindelöf and converging ω_1-sequences, Fundamenta Mathematicae, **224** (2014) 205–218.

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