The Katowice Problem

Tá scéilín agam

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Easy exercise one

Exercise

Let X and Y be two sets and $f : X \to Y$ a bijection.
Make a bijection between $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

Solution: $A \mapsto f[A]$ does the trick.

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The hard exercise

Exercise

Let X and Y be two sets and $F : \mathcal{P}(X) \to \mathcal{P}(Y)$ a bijection.
Make a bijection between X and Y.

Solution: can't be done.

Really!?

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How can that be?

But, if we have sets with the same number of subsets then they have the same number of points.

For if $2^m = 2^n$ then $m = n$.
True, for natural numbers m and n.

But that was not (really) the question.
The proof for m and n does not produce a bijection.
It does not use bijections at all.

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On to infinity

We have a scale to measure sets by: $\aleph_0$, $\aleph_1$, $\aleph_2$, $\aleph_3$ ...
$\aleph_0$ refers to countable.
$\aleph_1$ refers to the 'next' infinity
and so on ...

I teach this stuff every Friday afternoon in HG-10A33

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On to infinity

Remember Cantor's Continuum Hypothesis?
It says: $2^{\aleph_0} = \aleph_1$: the number of subsets of $\mathbb{N}$ is the smallest possible uncountable infinity.

When Cohen showed that the Continuum Hypothesis is unprovable, his method actually showed that $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ does not lead to contradictions.

This is a situation with a bijection between $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ but no bijection between X and Y.

Easy exercise two

Exercise

Let X and Y be two sets and $F : \mathcal{P}(X) \to \mathcal{P}(Y)$ a bijection that is also an isomorphism for the relation $\subseteq$.
Make a bijection between X and Y.

Solution: if $x \in X$ then $\{x\}$ is an atom (nothing between it and $\emptyset$), *hence* so is $F(\{x\})$.
But then $F(\{x\}) = \{y\}$ for some (unique) $y \in Y$.
There's your bijection.

Some algebra

We can consider $\mathcal{P}(X)$ as a group, or a ring.

Addition: symmetric difference
Multiplication: intersection

A $\subseteq$-isomorphism is also a ring-isomorphism.

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There is a nice ideal in the ring $\mathcal{P}(X)$:
the ideal, *fin*, of finite sets.

You can see where this is going . . .

The problem

The Katowice Problem

Let X and Y be sets and assume $\mathcal{P}(X)/\mathit{fin}$ and $\mathcal{P}(Y)/\mathit{fin}$ are ring-isomorphic.
Is there a bijection between X and Y?

Equivalently ...
If the Banach algebras $\ell^\infty(X)/c_0$ and $\ell^\infty(Y)/c_0$ are isomorphic must there be a bijection between X and Y?

Equivalently ...

The problem

... the original version

The Katowice Problem

If X^* and Y^* are homeomorphic must X and Y have the same cardinality.

Our sets carry the discrete topology and $X^* = \beta X \setminus X$, where βX is the Čech-Stone compactification.

Actually: X^* is also the structure space of $\ell^\infty(X)/c_0$ and the maximal-ideal space of $\mathcal{P}(X)/\mathit{fin}$.
So it all hangs together.

A bit of notation

We consider cardinals only: $\omega_0, \omega_1, \omega_2, \ldots$

In general A^* denotes the coset (equivalence class) in the $\mathcal{P}(\kappa)/\mathit{fin}$ that we are interested in.

Note: $A^* = B^*$ iff A and B differ by a finite set
$A^* \cdot B^* = 0$ iff $A \cap B$ is finite
and so on

Two results

Theorem (Frankiewicz 1977)

The minimum cardinal κ (if any) such that $\mathcal{P}(\kappa)/\mathit{fin}$ is isomorphic to $\mathcal{P}(\lambda)/\mathit{fin}$ for some $\lambda > \kappa$ must be ω_0.

Theorem (Balcar and Frankiewicz 1978)

$\mathcal{P}(\omega_1)/\mathit{fin}$ and $\mathcal{P}(\omega_2)/\mathit{fin}$ are not isomorphic.

Assume there are κ and λ ...

Let κ be minimal such that there is $\lambda > \kappa$ for which $\mathcal{P}(\kappa)/\mathit{fin}$ and $\mathcal{P}(\lambda)/\mathit{fin}$ are isomorphic.

Proposition

If $\kappa < \mu < \lambda$ then $\mathcal{P}(\kappa)/\mathit{fin}$ and $\mathcal{P}(\mu)/\mathit{fin}$ are isomorphic.

Proof.

Let $h : \mathcal{P}(\lambda)/\mathit{fin} \to \mathcal{P}(\kappa)/\mathit{fin}$ be a isomorphism and take $A \subseteq \kappa$ such that $A^* = h(\mu^*)$.
Note: $|A| < \mu$, so by minimality of κ we must have $|A| = \kappa$. $\square$

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Assume κ is the minimal ...

Proposition

$\kappa = \omega$

Proof.

Let $h : \mathcal{P}(\kappa)/\mathit{fin} \to \mathcal{P}(\kappa^+)/\mathit{fin}$ be a isomorphism.
For $\alpha < \kappa$ take $A_\alpha \subseteq \kappa^+$ such that $A_\alpha^* = h(\alpha^*)$ and let
$A = \bigcup_{\alpha<\kappa} A_\alpha$.
Note: $|A_\alpha| = |\alpha| < \kappa$ for all α, by minimality of κ, so $|A| \leqslant \kappa$. $\square$

Assume κ is the minimal ...

Proposition

$\kappa = \omega$

Proof, continued.

Take $B \subseteq \kappa$ such that $A^* = h(B^*)$, and so $(\kappa^+ \setminus A)^* = h((\kappa \setminus B)^*)$.
This implies $|\kappa \setminus B| = \kappa$.

But $\alpha^* \subseteq B^*$, which means $\alpha \setminus B$ is finite, for all α.
And so $|\kappa \setminus B| \leqslant \omega$. $\square$

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Scales

Let $\kappa > \omega_0$ and assume $\mathcal{P}(\omega_0)/\mathit{fin}$ and $\mathcal{P}(\kappa)/\mathit{fin}$ are isomorphic.

Consider $\omega_0 \times \kappa$ instead of κ and let $\gamma : \mathcal{P}(\omega_0 \times \kappa)/\mathit{fin} \to \mathcal{P}(\omega_0)/\mathit{fin}$ be a isomorphism.

Let $V_n = \{n\} \times \kappa$ and choose $v_n \subseteq \omega_0$ such that $v_n^* = \gamma(V_n^*)$.

We may rearrange the v_n to make them disjoint and even assume $v_n = \{n\} \times \omega_0$ for all n.

Scales

For $\alpha < \kappa$ let $E_\alpha = \omega_0 \times [\alpha, \kappa)$ and take $e_\alpha \subseteq \omega \times \omega$ such that $e_\alpha^* = \gamma(E_\alpha^*)$.
Define $f_\alpha : \omega \to \omega$ by

$$f_\alpha(n) = \min\{k : \langle n, k\rangle \in e_\alpha\}$$

Note: $f_\alpha \leqslant^* f_\beta$ if $\alpha < \beta$, i.e., $\{n : f_\alpha(n) > f_\beta(n)\}$ is finite.
For every $f : \omega \to \omega$ there is an α such that $f \leqslant^* f_\alpha$.

I.e., $\langle f_\alpha : \alpha < \kappa\rangle$ is a κ-scale.

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Scales

Assume $\mathcal{P}(\omega_1)/\mathit{fin}$ and $\mathcal{P}(\omega_2)/\mathit{fin}$ are isomorphic.

Then $\mathcal{P}(\omega_0)/\mathit{fin}$ and $\mathcal{P}(\omega_1)/\mathit{fin}$ must also be isomorphic.

But then we'd have an ω_1-scale and an ω_2-scale and hence a contradiction.

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Consequences

Corollary

If $\omega_1 \leqslant \kappa < \lambda$ then $\mathcal{P}(\kappa)/fin$ and $\mathcal{P}(\lambda)/fin$ are not isomorphic, and if $\omega_2 \leqslant \lambda$ then $\mathcal{P}(\omega_0)/fin$ and $\mathcal{P}(\lambda)/fin$ are not isomorphic.

So we are left with

Question

Are $\mathcal{P}(\omega_0)/fin$ and $\mathcal{P}(\omega_1)/fin$ ever isomorphic?

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So, what if they are isomorphic?

Easiest consequence: $2^{\aleph_0} = 2^{\aleph_1}$;

those are the respective cardinalities of $\mathcal{P}(\omega_0)/\mathit{fin}$ and $\mathcal{P}(\omega_1)/\mathit{fin}$

So CH implies 'no'.

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An ω_1-scale

Using the scales we get

$$\mathfrak{d} = \omega_1$$

And so $\mathrm{MA} + \neg\mathrm{CH}$ implies ‘no’.

A strong Q-sequence

In $\omega_0 \times \omega_1$ let $H_\alpha = \omega_0 \times \{\alpha\}$ and, for each α, choose $h_\alpha \subseteq \omega_0 \times \omega_0$ such that $\gamma(H_\alpha^*) = h_\alpha^*$.

$\{h_\alpha : \alpha < \omega_1\}$ is an almost disjoint family.
And a very special one at that.

Given $x_\alpha \subseteq h_\alpha$ for each α there is x such that $x \cap h_\alpha =^* x_\alpha$ for all α.

Basically $x^* = \gamma(X^*)$, where X is such that $(X \cap H_\alpha)^* = \gamma^{\leftarrow}(x_\alpha^*)$ for all α.

Such *strong Q-sequences* exist consistently (Steprāns).

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Even better (or worse?)

It is consistent to have

- $\mathfrak{d} = \omega_1$
- a strong Q-sequence
- $2^{\aleph_0} = 2^{\aleph_1}$

simultaneously (David Chodounsky).

(Actually second implies third.)

An automorphism of $\mathcal{P}(\omega_0)/\mathit{fin}$

Work with the set $D = \mathbb{Z} \times \omega_1$ — so now $\gamma : D^* \to \mathcal{P}(\omega_0)/\mathit{fin}$.

Define $\Sigma : D \to D$ by $\Sigma(n, \alpha) = \langle n + 1, \alpha \rangle$.

Then $\tau = \gamma \circ \Sigma^* \circ \gamma^{-1}$ is an automorphism of $\mathcal{P}(\omega_0)/\mathit{fin}$.

In fact, τ is non-trivial, i.e., there is no bijection $\sigma : a \to b$ between cofinite sets such that $\tau(x^*) = \sigma[x \cap a]^*$ for all subsets x of ω

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Light reading

Website: `fa.its.tudelft.nl/~hart`

K. P. Hart,
De ContinuumHypothese, Nieuw Archief voor Wiskunde, 10, nummer 1, (2009), 33–39

D. Chodounsky, A. Dow, K. P. Hart and H. de Vries
The Katowice problem and autohomeomorphisms of ω^*,
(arXiv e-print 1307.3930)

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