Compact spaces with a $\mathbb{P}$-diagonal

Tá scéilín agam

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$\mathbb{P}$-domination

Definition

A space, X, is $\mathbb{P}$-*dominated* (stop giggling) if there is a cover $\{K_f : f \in \mathbb{P}\}$ of X by compact sets such that $f \leqslant g$ (pointwise) implies $K_f \subseteq K_g$.

We call $\{K_f : f \in \mathbb{P}\}$ a $\mathbb{P}$-dominating cover.

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$\mathbb{P}$-diagonal

Definition

A space, X, has a $\mathbb{P}$-*diagonal* if the complement of the diagonal in X^2 is $\mathbb{P}$-dominated.

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Origins

Geometry of topological vector spaces (Cascales, Orihuela); $\mathbb{P}$-domination yields metrizability for compact subsets.

A compact space with a $\mathbb{P}$-diagonal is metrizable if it has countable tightness (no extra conditions if $\mathrm{MA}(\aleph_1)$ holds). (Cascales, Orihuela, Tkachuk).

Question

So, question: are compact spaces with $\mathbb{P}$-diagonals metrizable?

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Yes if CH (Dow, Guerrero Sánchez).

Two important steps in that result: a compact space with a $\mathbb{P}$-diagonal

- *does not* map onto $[0,1]^{\mathfrak{c}}$, ever
- *does* map onto $[0,1]^{\omega_1}$, when it has uncountable tightness

The answer

Theorem

Every compact space with a $\mathbb{P}$-diagonal is metrizable.

Proof.

No compact space with a $\mathbb{P}$-diagonal maps onto $[0,1]^{\omega_1}$. $\square$

How does that work?

BIG sets

We work with the Cantor cube 2^{ω_1}.

We call a closed subset, Y, of 2^{ω_1} BIG if there is a δ in ω_1 such that $\pi_\delta[Y] = 2^{\omega_1 \setminus \delta}$. ($\pi_\delta$ projects onto $2^{\omega_1 \setminus \delta}$)

Combinatorially: a closed set Y is BIG if there is a δ such that for every $s \in \mathrm{Fn}(\omega_1 \setminus \delta, 2)$ there is $y \in Y$ such that $s \subseteq y$.

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A nice property of BIG sets.

Proposition

A closed set is big if and only if there are a $\delta \in \omega_1$ and $\rho \in 2^\delta$ such that $\{x \in 2^{\omega_1} : \rho \subseteq x\} \subseteq Y$.

Of course 2^{ω_1} is $\mathbb{P}$-dominated: take $K_f = 2^{\omega_1}$ for all $f \in \mathbb{P}$.

Here is a Baire category-like result for 2^{ω_1}.

Theorem

If $\{K_f : f \in \mathbb{P}\}$ is a $\mathbb{P}$-dominating cover of 2^{ω_1} then some K_f is BIG.

The proof, case 1

$\mathfrak{d} = \aleph_1$: straightforward construction of a point not in $\bigcup_f K_f$ if we assume no K_f is BIG, using a cofinal family of $\aleph_1$ many K_f's.

The proof, case 3

$\mathfrak{b} > \aleph_1$: find there are $\aleph_1$ many $s \in \mathrm{Fn}(\omega_1, 2)$ and for each there are many $h \in \mathbb{P}$ such that $s \subseteq y$ for some $y \in K_h$.

We cleverly found $\aleph_1$ many h's such that each $\leqslant^*$-upper bound, f, for this family has a BIG K_f.

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The proof, case 2

$\mathfrak{d} > \mathfrak{b} = \aleph_1$: this is the trickiest one.
We borrow

Theorem (Todorčević)

If $\mathfrak{b} = \aleph_1$ then 2^{ω_1} has a subset X of cardinality $\aleph_1$ and such that every uncountable $A \subseteq X$ has a countable subset D such that $\pi_\delta[D]$ is dense in $2^{\omega_1 \setminus \delta}$ for some δ.

This yields another set of $\aleph_1$ many h's; the special properties of X ensure: if f is not dominated by any one of the h's then K_f is BIG.

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The final step: assume X has a $\mathbb{P}$-diagonal and a continuous map onto $[0,1]^{\omega_1}$.

Then we have a closed subset Y with a $\mathbb{P}$-diagonal and a continuous map φ of Y onto 2^{ω_1}.

Then we find closed sets $Y_0 \supset Y_1 \supset \cdots$ and points $y_n \in Y_n \setminus Y_{n+1}$ such that $\varphi[Y_n]$ is always BIG and (ultimately) one f such that $\bigcup_n(\{y_n\} \times Y_{n+1}) \subseteq K_f$.

For every accumulation point, y, of $\langle y_n : n \in \omega\rangle$ we'll have $\langle y, y\rangle \in K_f$, a contradiction.

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Question

Cascales, Orihuela and Tkachuk also asked if a compact space with a $\mathbb{P}$-diagonal would have a small diagonal (answer: yes); this would imply metrizability.

I'd like to turn that around: does a space with a small diagonal have a $\mathbb{P}$-diagonal?
This would settle the metrizability question for spaces with a small diagonal.

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Light reading

Website: `fa.its.tudelft.nl/~hart`

Alan Dow and Klaas Pieter Hart,
Compact spaces with a $\mathbb{P}$-diagonal, Indagationes Mathematicae, **27** (2016), 721–726.

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