The Katowice Problem

Tá scéilín agam

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Easy exercise one

Exercise

Let X and Y be two sets and $f : X \to Y$ a bijection.
Make a bijection between $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

Solution: $A \mapsto f[A]$ does the trick.

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The hard exercise

Exercise

Let X and Y be two sets and $F : \mathcal{P}(X) \to \mathcal{P}(Y)$ a bijection.
Make a bijection between X and Y.

Solution: can't be done.

Really!?

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How can that be?

But, if we have sets with the same number of subsets then they have the same number of points.

For if $2^m = 2^n$ then $m = n$.
True, for natural numbers m and n.

But that was not (really) the question.
The proof for m and n does not produce a bijection.
It does not use bijections at all.

On to infinity

We have a scale to measure sets by: $\aleph_0$, $\aleph_1$, $\aleph_2$, $\aleph_3$, ...
$\aleph_0$ refers to countable.
$\aleph_1$ refers to the ‘next’ infinity
and so on ...

I teach this stuff every Friday afternoon in SP 904 (C1.112)

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On to infinity

Remember Cantor's Continuum Hypothesis?
It says: $2^{\aleph_0} = \aleph_1$: the number of subsets of $\mathbb{N}$ is the smallest possible uncountable infinity.

When Cohen showed that the Continuum Hypothesis is unprovable, his method actually showed that $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ does not lead to contradictions.

This is a situation with a bijection between $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ but no bijection between X and Y.

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Easy exercise two

Exercise

Let X and Y be two sets and $F : \mathcal{P}(X) \to \mathcal{P}(Y)$ a bijection that is also an isomorphism for the relation $\subseteq$.
Make a bijection between X and Y.

Solution: if $x \in X$ then $\{x\}$ is an atom (nothing between it and $\emptyset$), *hence* so is $F(\{x\})$.
But then $F(\{x\}) = \{y\}$ for some (unique) $y \in Y$.
There's your bijection.

Some algebra

We can consider $\mathcal{P}(X)$ as a group, or a ring.

Addition: symmetric difference
Multiplication: intersection

A $\subseteq$-isomorphism is also a ring-isomorphism.

There is a nice ideal in the ring $\mathcal{P}(X)$:
the ideal, *fin*, of finite sets.

You can see where this is going . . .

The problem

The Katowice Problem

Let X and Y be sets and assume $\mathcal{P}(X)/\mathit{fin}$ and $\mathcal{P}(Y)/\mathit{fin}$ are ring-isomorphic.
Is there a bijection between X and Y?

Equivalently ...
If the Banach algebras $\ell^\infty(X)/c_0$ and $\ell^\infty(Y)/c_0$ are isomorphic must there be a bijection between X and Y?

Equivalently ...

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The problem

... the original version

The Katowice Problem

If X^* and Y^* are homeomorphic must X and Y have the same cardinality.

Our sets carry the discrete topology and $X^* = \beta X \setminus X$, where βX is the Čech-Stone compactification.

Actually: X^* is also the structure space of $\ell^\infty(X)/c_0$ and the maximal-ideal space of $\mathcal{P}(X)/\mathit{fin}$.
So it all hangs together.

Two results

Theorem (Frankiewicz 1977)

The minimum cardinal κ (if any) such that $\mathcal{P}(\kappa)/\mathit{fin}$ is isomorphic to $\mathcal{P}(\lambda)/\mathit{fin}$ for some $\lambda > \kappa$ must be ω_0.

Theorem (Balcar and Frankiewicz 1978)

$\mathcal{P}(\omega_1)/\mathit{fin}$ and $\mathcal{P}(\omega_2)/\mathit{fin}$ are not isomorphic.

Consequences

Corollary

If $\omega_1 \leqslant \kappa < \lambda$ then $\mathcal{P}(\kappa)/\mathit{fin}$ and $\mathcal{P}(\lambda)/\mathit{fin}$ are not isomorphic, and if $\omega_2 \leqslant \lambda$ then $\mathcal{P}(\omega_0)/\mathit{fin}$ and $\mathcal{P}(\lambda)/\mathit{fin}$ are not isomorphic.

So we are left with

Question

Are $\mathcal{P}(\omega_0)/\mathit{fin}$ and $\mathcal{P}(\omega_1)/\mathit{fin}$ ever isomorphic?

Why ‘ever’?

The Continuum Hypothesis implies that $\mathcal{P}(\omega_0)/\mathit{fin}$ and $\mathcal{P}(\omega_1)/\mathit{fin}$ are not isomorphic?

So, we can not prove that they are isomorphic.

But, can we prove they they are not isomorphic?

The “are they ever” translates to:
is there a model of Set Theory where $\mathcal{P}(\omega_0)/\mathit{fin}$ and $\mathcal{P}(\omega_1)/\mathit{fin}$ are isomorphic?

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Consequences

We want "$\mathcal{P}(\omega_0)/\mathit{fin}$ and $\mathcal{P}(\omega_1)/\mathit{fin}$ are isomorphic" to be false.

We have many consequences.

But not yet $0 = 1$.

Here's a nice one . . .

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An automorphism of $\mathcal{P}(\omega_0)/\mathit{fin}$

Work with the set $D = \mathbb{Z} \times \omega_1$ — so we assume $\gamma : \mathcal{P}(D)/\mathit{fin} \to \mathcal{P}(\omega_0)/\mathit{fin}$ is an isomorphism.

Define $\Sigma : D \to D$ by $\Sigma(n, \alpha) = \langle n + 1, \alpha \rangle$.

Then $\tau = \gamma \circ \Sigma^* \circ \gamma^{-1}$ is an automorphism of $\mathcal{P}(\omega_0)/\mathit{fin}$.

In fact, τ is non-trivial, i.e., there is no bijection $\sigma : a \to b$ between cofinite sets such that $\tau(x^*) = \sigma[x \cap a]^*$ for all subsets x of ω

Light reading

Website: `fa.its.tudelft.nl/~hart`

K. P. Hart,
De ContinuumHypothese, Nieuw Archief voor Wiskunde, 10, nummer 1, (2009), 33–39

D. Chodounsky, A. Dow, K. P. Hart and H. de Vries
The Katowice problem and autohomeomorphisms of ω^*,
(arXiv e-print 1307.3930)

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