Gary and me

Tá scéilín agam

K. P. Hart

Faculty EEMCS
TU Delft

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M_3 and M_2

Some 40 years ago: Maarten Maurice taught a course on generalized metric spaces.

Naturally M_1-, M_2- and M_3-spaces made their appearance.

And Maarten mentioned that someone named Gruenhage had shown that M_3 implies M_2.

(The pronuciation of the 'hage' part of the name was a bit unclear to us.)

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"Can I read that?"

I got a copy of Gary's paper from Maarten and I read it.

And that lead to my first seminar talk ever.

One member of the audience then is in the audience now as well.

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Just one question . . .

How would you render this notation in $\TeX$?

We shall make frequent use of the following notation. If $F \subset V \subset X$, where F is closed and V is open in the stratifiable space X, let $V_F = D(F, X - V)$, where D is a monotone normality operator on X. If $n \in N$, and $V_F n$ is defined, let $V_F n+1 = D(F, X - V_F n)$. If p is a point in V, let $V_p n = V_{\{p\}} n$. Note that if $F \subset G$ and $V \subset W$, and W_G is defined, then $V_F n \subset W_G n$ for all $n \in N$.

Would that be $V_F n$, or V_{F^n}, or rather $V_F(n)$?

A common hobby

I occasionally send in solutions to Monthly problems.

In 2000, out of the blue, an email from Gary:
"Congratulations on getting a solution published."
(It took Monthly issues longer to get to Europe than to Auburn, I guess.)

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GCHQ

This is GCHQ (the UK's NSA).

They have a file on Gary

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Gary's problem

Here's why.

10799. *Proposed by Curtis Herink, Mercer University, Macon, GA, and Gary Gruenhage, Auburn University, Auburn, AL.* Let κ and λ be infinite cardinals with $\kappa > \lambda$. Let X be a topological space with at least κ open sets. Show that if every open cover of X containing exactly κ open sets has a finite subcover, then every open cover of X containing exactly λ open sets has a finite subcover.

Monthly: volume 107, No. 4 (April, 2000) p. 367

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Gary's problem

One year later ...

Finite Subcovers for Open Covers with Exact Cardinality

10799 [2000, 367]. *Proposed by Curtis Herink, Mercer University, Macon, GA, and Gary Gruenhage, Auburn University, Auburn, AL.* Let κ and λ be infinite cardinals with $\kappa > \lambda$. Let X be a topological space with at least κ open sets. Show that if every open cover of X containing exactly κ open sets has a finite subcover, then every open cover of X containing exactly λ open sets has a finite subcover.

Solution by the GCHQ Problems Group, Cheltenham, U. K. Call an open set with at least κ open subsets *large*, and one with fewer than κ open subsets *small*. If $a_1, a_2, \ldots, a_n$ are small open sets, with, respectively, $\mu_1, \mu_2, \ldots, \mu_n$ open subsets, where each $\mu_i < \kappa$, then $\bigcup_{i=1}^n a_i$ has at most $\prod_{i=1}^n \mu_i < \kappa$ open subsets. Hence a finite union of small sets is small.

Monthly: volume 108, No. 3 (March, 2000) p. 278

A recent solution

Almost a year ago:

A Binary Operation Whose Closed Sets Form a Chain

11813 [2015, 76]. *Proposed by Greg Oman, University of Colorado–Colorado Springs, Colorado Springs, CO.* Let X be a set, and let $*$ be a binary operation on X (that is, a function from $X \times X$ to X). Prove or disprove: there exists an uncountable set X and a binary operation $*$ on X such that for any subsets Y and Z of X that are closed under $*$, either $Y \subseteq Z$ or $Z \subseteq Y$.

Solution by Gary Gruenhage, Auburn University, Auburn, AL. Such sets X and binary operations $*$ do exist, and $*$ can even be commutative. Let X be the set of count-

Monthly: volume 123, No. 9 (November, 2016) p. 947

About the solution

Gary's treatment:

- a commutative operation on ω_1
- cannot be associative
- cannot be done on ω_2

I was an also-ran

Another problem

We have a new race on:

11943. *Proposed by Keith Kearnes, University of Colorado, Boulder, CO, and Greg Oman, University of Colorado, Colorado Springs, CO.* Let X be a set, and let $\mathcal{F}$ be a collection of functions f from X into X. A subset Y of X is *closed* under $\mathcal{F}$ if $f(y) \in Y$ for all $y \in Y$ and f in $\mathcal{F}$. With the axiom of choice given, prove or disprove: There exists an uncountable collection $\mathcal{F}$ of functions mapping $\mathbb{Z}^+$ into $\mathbb{Z}^+$ such that
(a) every proper subset of $\mathbb{Z}^+$ that is closed under $\mathcal{F}$ is finite, and
(b) for every $f \in \mathcal{F}$, there is a proper infinite subset Y of $\mathbb{Z}^+$ that is closed under $\mathcal{F}\setminus\{f\}$.

Monthly: volume 123, No. 10 (December, 2016) p. 1050
May the best solution win.

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Small diagonals

Definition (Van Douwen, Hušek, Zhou)

A space, X, has a *small diagonal* if every uncountable subset of $X^2 \setminus \Delta(X)$ has an uncountable subset whose closure is disjoint from $\Delta(X)$.

Hušek defined the negation: X has an ω_1-accessible diagonal if there is a sequence $\langle \langle x_\alpha, y_\alpha \rangle : \alpha \in \omega_1 \rangle$ that converges to $\Delta(X)$ and always $x_\alpha \neq y_\alpha$ of course.

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A sufficient condition

If $\Delta(X)$ is a G_δ-set then X has a small diagonal.
Hence, for example, metrizable spaces have small diagonals.

Elementary sequences

From now on Alan shares the blame for everything not attributed to Gary.

Many proofs of results on csD spaces (compact small Diagonal) work as follows.

Let X be compact; take a sequence $\langle M_\alpha : \alpha \in \omega_1 \rangle$ of countable elementary substructures with $X \in M_0$ — an *elementary sequence for* X.

Elementary sequences of pairs

We assume X to be embedded into $[0,1]^\kappa$, say $\kappa = w(X)$, so the following makes sense.

An *elementary sequence of pairs for X* is a sequence $\langle \{x_\alpha, y_\alpha\} : \alpha \in \omega_1 \rangle$ such that, always,

- $x_\alpha \restriction M_\alpha = y_\alpha \restriction M_\alpha$,
- $x_\alpha \neq y_\alpha$, and
- $\{x_\alpha, y_\alpha\} \in M_{\alpha+1}$.

One of the coordinate sequences may be constant.

Elementary sequences and small Diagonals

Gary: a compact space is csD iff every ω_1-sequence of pairs is ω_1-*separated*, i.e., there is an uncountable set A such that $\mathrm{cl}\{x_\alpha : \alpha \in A\}$ and $\mathrm{cl}\{y_\alpha : \alpha \in A\}$ are disjoint.

In fact: a compact space is csD iff every elementary sequence of pairs is ω_1-separated.

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small Diagonals and metrizability

Remember: a compact space is metrizable iff its diagonal is a G_δ-set.

Also: a compact space is metrizable iff every elementary sequence of pairs does not exist.

We are lead to ask

Is every csD space metrizable?

A sufficient condition

Gary: If X is csD and hereditarily Lindelöf (equivalently, perfectly normal) then X is metrizable.

Take an elementary sequence of pairs and $A \subseteq \omega_1$ uncountable.
Then $\{x_\alpha : \alpha \in A\}$ is Lindelöf.

Pick $\delta \in A$ such that $\{\alpha \in A : x_\alpha \in V\}$ is uncountable, for every basic neighbourhood V of x_δ.

$M_{\delta+1}$ contains a countable local base, $\mathcal{B}$, at x_δ.

For each member B of $\mathcal{B}$ and $\alpha > \delta$ we have $x_\alpha \in B$ iff $y_\alpha \in B$.

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A sufficient condition

Let us squeeze whatever we can out of that proof, put
$M = \bigcup_\alpha M_\alpha$.

It suffices that X be first-countable and $X \cap M$ be Lindelöf:
we don't need x_δ, just an x in $X \cap M$.

For the latter Lindelöfness of $X \cap M$ is enough.

We do need a countable local base at x to make the last part work.

A sufficient condition

It suffices that $X \cap M$ be Lindelöf for just one elementary sequence. For then we can prove that our csD space X is first-countable.

If X is not then some $x \in X \cap M_0$ does not have a countable local base.

Hence we can choose $x_\alpha \in M_{\alpha+1}$ such that $x_\alpha \neq x$, but $x_\alpha \restriction M_\alpha = x \restriction M_\alpha$.

Let $A \subseteq \omega_1$ be uncountable and let $y \in M$ be a complete accumulation point of $\{x_\alpha : \alpha \in A\}$.

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A sufficient condition

It follows that $y \restriction M_\alpha = x \restriction M_\alpha$ for all α and hence $x \restriction M = y \restriction M$.

By elementarity: $x = y$.

The sequence $\langle x_\alpha : \alpha \in \omega_1 \rangle$ converges to x.

Contradiction.

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Thanks

Thanks for the fun Gary.

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