Borel circle-squaring

Tá scéilín agam

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The problem

Fundamenta Mathematicae, 1925:

38) Un carré et un cercle dont les aires sont égales peuvent-ils être décomposés en un nombre fini de sous-ensembles disjoints respectivement congruents?

Problème de M. Tarski.

No restrictions on the nature of the pieces.

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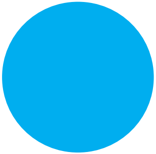
The problem

So: change  into 

by dividing it into finitely many subsets and rearranging them, using rigid motions.

The Riemann Mapping Theorem does not count.

Compass and straightedge

First attempt would be: compass and straightedge.

Conspicuously absent in Euclid's *Elements*.

One would have to construct a line segment of length π.

Archimedes: the disc has the same area as a triangle with base π and height 1.

Compass and straightedge

We now know why this would not work.

Every constructible number is algebraic (of degree a power of 2).

Lindemann (1882): π is transcendental.

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Banach-Tarski

We all know the Banach-Tarski phenomenon.

One can decompose the unit ball into finitely many subsets;
rearrange these subsets using rigid motions;
and reassemble them into two unit spheres.

The plane

There is no Banach-Tarski phenomenon in the plane.

Why?

Because . . .

The plane

Theorem

There is a finitely additive and isometry-invariant extension of (planar) Lebesgue-measure to the full power set of $\mathbb{R}^2$.

So, in the plane the condition 'of equal measure' is important.

Scissors maybe?

Dubins, Hirsch, and Karush: not with scissors.

You cannot make a jig-saw puzzle out of the disc that can also be laid out as a square.

Jig-saw puzzle:
the pieces are Jordan-curves-plus-their-interior-domains.
We allow overlap on the boundaries.

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Scissors maybe?

Why?

An informative picture on the blackboard.

Ugly sets

Laczkovich (1990): it can be done using translations only.

But it needs the Axiom of Choice and the pieces are not very nice (not measurable).

Nicer sets

Marks and Unger (2016): it can be done using translations only.

And with Borel measurable pieces.

In fact the complexity of the pieces is not higher than five.

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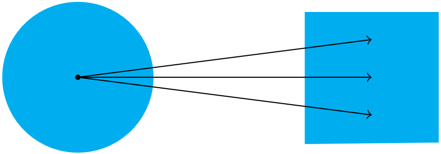
How to translate

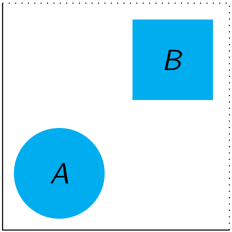


Where should that point go?

Finding veeery many candidates

We work in the unit square $S = [0, 1)^2$.

Let A and B be scaled versions in S of our disc and square respectively.



Finding veeery many candidates

Very technical lemma

For almost every $\boldsymbol{u} \in S^5$ there are $\varepsilon > 0$ and $M > 0$ such that for every $\boldsymbol{x} \in S$ and $N > 0$

$$D\bigl(F_N(\boldsymbol{x}, \boldsymbol{u}), X)\bigr) \leqslant M \cdot N^{-1-\varepsilon}$$

for both $X = A$ and $X = B$.

What is all that?

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Finding veeery many candidates

- $F_N(\boldsymbol{x}, \boldsymbol{u})$ is the set of all $\mathbb{Z}$-linear combinations

$$\boldsymbol{x} + n_1\boldsymbol{u}_1 + n_2\boldsymbol{u}_2 + n_3\boldsymbol{u}_3 + n_4\boldsymbol{u}_4 + n_5\boldsymbol{u}_5$$

  with integer n_i such that $0 \leqslant n_i < N$ for all i.
  These are reduced modulo 1 (to get back into S).
- $D(F, X)$ is the 'discrepancy' of F relative to X:

$$D(F, X) = \left| \frac{|F \cap X|}{|F|} - \lambda(X) \right|$$

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Finding veeery many candidates

We take one such $\boldsymbol{u}$ and use it to define a directed graph on S.
From every $\boldsymbol{x}$ there emanate five arrows: $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{u}_1$, $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{u}_2$, $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{u}_3$, $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{u}_4$, and $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{u}_5$.
(Everything still reduced modulo 1.)

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Finding veeery many candidates

Very technical lemma

For almost every $\boldsymbol{u} \in S^5$ there are $\varepsilon > 0$ and $M > 0$ such that for every $\boldsymbol{x} \in S$ and $N > 0$

$$D\bigl(F_N(\boldsymbol{x}, \boldsymbol{u}), X)\bigr) \leqslant M \cdot N^{-1-\varepsilon}$$

for both $X = A$ and $X = B$.

So this Lemma says that we can take, uniformly, graph-neighbourhoods of every point that approximate the measures of A and B as well as we please.

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How to choose?

First choose a suitably large N to have the corresponding graph-neighbourhoods give good approximations of A and B.

It is then possible to define a Borel-measurable flow φ on the graph such that

- every point of A produces 1
- every point of B receives 1
- all other points of S produce/receive 0

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How to choose?

Via this flow you can construct a Borel-measurable bijection $f : A \to B$ such that for every $\boldsymbol{x} \in A$ the value $f(\boldsymbol{x})$ is in the graph-neighbourhood $F_N(\boldsymbol{x}, \boldsymbol{u})$.

But now we have 5^N translations $\{T_k : k = 1, \ldots, 5^N\}$ such that for every $\boldsymbol{x}$ there is a k with $f(\boldsymbol{x}) = T_k(\boldsymbol{x})$.

There is your partition: $A_k = \{\boldsymbol{x} \in A : f(\boldsymbol{x}) = T_k(\boldsymbol{x})\}$.

During a lecture in February Andrew Marks mentioned the following upper bound for the number of pieces: 10^{220}.

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Light reading

Andrew S. Marks and Spencer T. Unger
Borel Circle Squaring, https://arxiv.org/abs/1612.05833