The Axiom of Choice

Tá scéilín agam

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Sets

End of the 19th century: Georg Cantor developed Set Theory.

Important instruments: well-orders and ordinal numbers.

Well-order

A linear order, $\prec$, of a set X is a well-order is every nonempty subset of X has a minimum
(with respect to $\prec$).

Well-orders are useful because they facilitate induction and recursion.

Ordinal numbers

Cantor's definition: an ordinal number is the 'order type' of a well-ordered set.

Problem: 'order type' has a rather vague definition.

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Ordinal numbers

Modern definition: a transitive set that is well-ordered by $\in$.

A bit unusual perhaps but unambiguous.

Every well-ordered set is isomorphic with exactly one such ordinal number.

That gives an unambiguous definition of order type.

The big question

Can every set be well-ordered?

Cantor thought so, but he did not have a proof.

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From Hilbert's First Problem

The question now arises whether the totality of all numbers may not be arranged in another manner so that every partial assemblage may have a first element, i.e., whether the continuum cannot be considered as a well ordered assemblage — a question which Cantor thinks must be answered in the affirmative. It appears to me most desirable to obtain a direct proof of this remarkable statement of Cantor's, perhaps by actually giving an arrangement of numbers such that in every partial system a first number can be pointed out.

Bulletin of the American Mathematical Society **8** (1901), 437–479

The Well-ordering Theorem

Stelling (Zermelo, 1904)

Every set, M, can be well-ordered.

The first step in the proof:
take a map γ that assigns to every nonempty subset N of M a 'chosen' element $\gamma(N)$ of N.

The Well-ordering Theorem

Justification:

the number of such assignments is equal to the product $\prod \mathfrak{n}$ of the powers of those sets N
and therefore not equal to 0.

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The Well-ordering Theorem

The problem with this argument?

The 'therefore' is circular: the statement
"the product of non-zero powers is again non-zero"
is a reformulation of
"there is such an assignment γ"

Aside

In the algebra of $\mathbb{R}$ we have:

if $x_i \neq 0$ for $i = 1, 2, \ldots, n$ then $\prod_{i=1}^{n} x_i \neq 0$

you can *prove* this from the axioms for $\mathbb{R}$.

Aside

The definition of $\prod_i \mathfrak{n}_i$ is:

the power of the product $\prod_i N_i$

Completely different from the multiplication in $\mathbb{R}$.

Generalisation from $\mathbb{R}$ to powers is not allowed.

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From theorem to axiom

In 1908 Zermelo gave an axiomatisation of Set Theory.

"to be able to prove that *a product of sets is empty if and only at least one factor is empty* we need another axiom:"

"let it always be possible *to choose* from every element M, N, R $\ldots$ of T a unique element m, n, r, $\ldots$ and to collect these in a set S_1"

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Bertrand Russell, 1905

On some difficulties in the theory of transfinite numbers and order types.

Nice article, with *Russell's Paradox*: for $R = \{x : x \notin x\}$ we have: $R \in R$ if and only if $R \notin R$.

And also a section on *Zermelo's Axiom*.

Bertrand Russell, 1905

Given $\aleph_0$ pairs of boots, let it be required to prove that the number of boots is even.

How?

This will be the case if all the boots can be divided into two classes which are mutually similar.

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Bertrand Russell, 1905

If each pair has the right and left boots different, we need only put all the right boots in one class, and all the left boots in another:

the class of right boots is similar to the class of left boots and our problem is solved.

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Bertrand Russell, 1905

But, if the right and left boots in each pair are indistinguishable, we cannot discover any property belonging to exactly half the boots.

Hence we cannot divide the boots into two equal parts, and we cannot prove that the number of them is even.

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Bertrand Russell, 1905

If the number of pairs were finite, we could simply choose one out of each pair;

but we cannot choose one out of each of an infinite number of pairs unless we have a *rule* of choice, and in the present case no rule can be found.

Note: there was a time when there was not always a difference between left and right shoes.

Modern versions of this argument use shoes and socks.

Continuity

Everyone knows this theorem:

Stelling

A function $f : D \to \mathbb{R}$ is continuous at p if and only if for every sequence $\langle x_n \rangle_n$ in D with limit p we have $\lim_n f(x_n) = f(p)$.

Proof.

From left to right: not hard (follow your nose). □

Continuity

Proof.

From right to left: *suppose not* and let $\varepsilon > 0$ be such that for every $\delta > 0$ there is an $x \in D$ for which $|x - p| < \delta$ end $\bigl|f(x) - f(p)\bigr| \geqslant \varepsilon$.

For every n choose such an x_n for $\delta = 2^{-n}$ and there is we have a sequence $\langle x_n \rangle_n$ with $\lim_n x_n = p$ whereas $\bigl|f(x_n) - f(p)\bigr| \geqslant \varepsilon$ for all n $\square$

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Continuity

That was a blatant application of the Axiom of Choice:

we proved: for every n the set

$$X_n = \{x \in D : |x - p| < 2^{-n} \text{ en } \bigl|f(x) - f(p)\bigr| \geqslant \varepsilon\}$$

is nonempty

we used: $\prod_{n\in\mathbb{N}} X_n$ is nonempty

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Is it true?

Good question. The reactions diverged.

On the one hand: yes, of course! In naïve Set Theory such a simultaneous choice was deemed possible

On the other hand: well, . . . , how do you make such a simultaneous choice? (see Russell).

(There were stronger reactions than just "well . . . ".)

Is it true?

But these reactions tell us nothing about the truth of the Axiom of Choice.

What is 'true'?

Truth is one of the hardest things to define.

Is $2 \times 2 = 0$ true? (Yes: modulo 4; no: in $\mathbb{R}$)
Truth is relative.

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Is it provable?

Provability is relative as well: what are the assumptions?

If I adopt the Axiom of Choice as an assumption then it becomes automatically provable.

I could adopt the Well-ordering Theorem as an assumption; then the Axiom of Choice become provable too.

Proof.

Take a well-ordering of the union of the sets en use the rule "take the minimum". □

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Gödel, 1940

Stelling

Based on Zermelo's axiomatisation of Set Theory the negation of the Axiom of Choice is not provable.

Phew!

Using the Axiom of Choice does not lead to contradictions.

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Cohen, 1963

Stelling

Based on Zermelo's axiomatisation of Set Theory the Axiom of Choice is not provable.

Russell and the others were right: the Axiom of Choice is really an extra assumption.

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Is the Axiom of Choice useful?

Yes. According to many.

Very soon the Axiom of Choice was used in Mathematics.
Sometimes it was announced explicitly:
"Under assumption of the Axiom of Choice . . . "

Later people stopped announcing it.

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Example: the Hahn-Banach Theorem

Stelling

Let G be a subspace of a normed space E and let $f : G \to \mathbb{R}$ be linear and continuous, then f has an extension $\bar{f} : E \to \mathbb{R}$ that satisfies

$$\sup\left\{\left|f(a)\right| : a \in G, \|a\| = 1\right\} = \sup\left\{\left|\bar{f}(x)\right| : x \in E, \|x\| = 1\right\}$$

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Example: The Hahn-Banach Theorem

From Banach's proof:

> On prouve ce théorème par induction transfinie en appliquant succesivement le théorème 1 aux éléments de l'ensemble $E - G$ (supposé bien ordonné).

Here 'théorème 1' refers to the possibility of extending such an f to a subspace obtained by adding just one vector to the given subspace.

Famous consequences

- Tychonoff's Theorem: every product of compact topological spaces is compact.
- Representation theorems:
  - every commutative Banach-algebra is of the form $C(K)$ with K compact
  - every Boolean algebra is isomorphic to a an algebra of sets
- The Hahn-Banach Theorem (see above)

Famous consequences

- Krull's theorem: every ring with 1 has maximal ideals.
- The Nielsen-Schreier theorem: every subgroup of a free group is free
- Every vector space (over any field) has a basis, and
- All bases of a vector space have the same number of elements
- If X is infinite then there is an injection $f : \mathbb{N} \to X$

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One more consequence

For any two sets A and B we have:
there is an injection $f : A \to B$ or there is an injection $g : B \to A$.

Cantor thought that should simply be true.

Hartogs proved that this implies the Well-ordering Theorem.

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Some strange/unwanted(?) consequences

Vitali (1905): There are non-measurable sets.

Banach and Tarski (1924): the unit ball in $\mathbb{R}^3$ can be decomposed into finitely many sets that can be reassembled into two copies of the unit ball.
(These sets are non-measurable as well.)

Equivalent statements

- The Well-ordering Theorem
- Zorn's Lemma: if in a partially ordered set every chain (linearly ordered subset) has an upper bound then that set has maximal elements
- Hausdorff's Maximality Principle: every partially ordered set has maximal chains
- Teichmüller-Tukey: ever family of sets that is of finite character has maximal elements (finite character: a set belongs to the family if and only if every finite subset is in the family)

Future problems?

The name of this Symposium is ‘Future Problems’.

Are there Future Problems about the Axiom of Choice?

Yes: every result produces questions:

“Does it need choice?”

Is there a ‘constructive’ version?

Axiom of Choice (the band)

Axiom of Choice: There is an exciting and profound artistic value in the mathematical principle, Axiom of Choice. The mathematician has the right to choose elements without explanation. In a world where everything must be explained, these choices are voluntary and do not need explanation.

(Liner notes to the album *Beyond Denial* (1994,1995).)

Light reading

Website: `fa.its.tudelft.nl/~hart`

Paul Howard and Jean E. Rubin,
Consequences of the axiom of choice,
American Mathematical Society (1998).
http://www.math.purdue.edu/~hrubin/JeanRubin/Papers/con

Thomas Jech,
The Axiom of Choice,
Dover Publications (2008).

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