Soft compactifications of $\mathbb{N}$

Tá scéilín agam

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Is each Parovichenko compact space homeomorphic to the remainder of a soft compactification of $\mathbb{N}$?

Ask Question

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Definition 1. A compactification $c\mathbb{N}$ of the discrete space $\mathbb{N}$ is called *soft* if for any disjoint sets $A, B \subset \mathbb{N}$ with $\bar{A} \cap \bar{B} \neq \emptyset$ there exists a homeomorphism $h : c\mathbb{N} \to c\mathbb{N}$ such that $h(x) = x$ for all $x \in c\mathbb{N} \setminus \mathbb{N}$ and the set $\{x \in A : h(x) \in B\}$ is infinite.

Definition 2. A compact Hausdorff space X is called *Parovichenko* (resp. *soft Parovichenko*) if X is homeomorphic to the remainder $c\mathbb{N} \setminus \mathbb{N}$ of some (soft) compactification $c\mathbb{N}$ of $\mathbb{N}$?

Remark 1. By a classical Parovichenko Theorem, each compact Hausdorff space of weight $\leq \aleph_1$ is Parovichenko. Hence, under CH a compact Hausdorff space is Parovichenko if and only if it has weight $\leq \mathfrak{c}$. By a result of Przymusinski, each perfectly normal compact space is Parovichenko. On the other hand, Bell constructed an consistent example of a first-countable compact Hausdorff space, which is not Parovichenko. More information and references on Parovichenko spaces can be found in this survey of Hart and van Mill (see §3.10).

Problem 1. Is each Parovichenko compact space soft Parovichenko?

Remark 2. The Stone-Cech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ is soft, but there are simple examples of compactifications which are not soft. A compactification $c\mathbb{N}$ of $\mathbb{N}$ is soft if for any disjoint sets $A, B \subset \mathbb{N}$ with $\bar{A} \cap \bar{B} \neq \emptyset$ there are sequences $\{a_n\}_{n\in\omega} \subset A$ and $\{b_n\}_{n\in\omega} \subset B$ that converge to the same point $x \in \bar{A} \cap \bar{B}$. This implies that a compactification $c\mathbb{N}$ is soft if the space $c\mathbb{N}$ is Frechet-Urysohn or has sequential square. This also implies that *each first-countable Parovichenko space is soft Parovichenko* (more generally, *a Parovichenko space X is soft Parovichenko if each point $x \in X$ has a neighborhood base of cardinality $< \mathfrak{p}$*).

Problem 2. Is each (Frechet-Urysohn) sequential Parovichenko space soft Parovichenko?

The following concrete version of Problem 1 describes an example of a Parovichenko space for which we do not know if it is soft Parovichenko.

Problem 3. Let X be a compact space that can be written as the union $X = A \cup B$ where A is homeomorphic to $\beta\mathbb{N} \setminus \mathbb{N}$, B is homeomorphic to the Cantor cube $\{0, 1\}^\omega$ and $A \cap B \neq \emptyset$. Is the space X soft Parovichenko?

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In larger print

A compactification $\gamma\mathbb{N}$ of $\mathbb{N}$ is soft if
whenever A and B are disjoint subsets of $\mathbb{N}$ with $\operatorname{cl} A \cap \operatorname{cl} B \neq \emptyset$
there is an autohomeomorphism h of $\gamma\mathbb{N}$
that is the identity on $\gamma\mathbb{N} \setminus \mathbb{N}$
and such that $h[A] \cap B$ is infinite.

Why?

Softness is a sufficient condition for a compactification to be the Higson corona of a finitary coarse space.

To pre-empt an obvious question:
no, I do not know why the word ‘soft’ was chosen.

Examples

The Čech-Stone compactification $\beta\mathbb{N}$ is soft ... vacuously
there are no disjoint subsets of $\mathbb{N}$ with disjoint closures ...

The one-point compactification $\alpha\mathbb{N} = \omega + 1$ is soft:
take a permutation h of ω with $h[A] = B$

Examples

If $\gamma\mathbb{N}$ is a metric compactification then it is soft.
If $x \in \operatorname{cl} A \cap \operatorname{cl} B$ then there are sequences $\langle a_n : n \in \omega \rangle$ and $\langle b_n : n \in \omega \rangle$ in A and B respectively that converge to x.
Define h on $\mathbb{N}$ by $h(a_n) = b_n$, $h(b_n) = a_n$, and $h(n) = n$ otherwise.

(Yes, yes, I know: Fréchet-Urysohn suffices . . .)

The question

"Is each Parovichenko compact space soft-Parovichenko?"

Translation

If X is compact Hausdorff and there is a compactification $\gamma\mathbb{N}$ of $\mathbb{N}$ such that $X = \gamma\mathbb{N} \setminus \mathbb{N}$ is there then
a soft compactification $\delta\mathbb{N}$ of $\mathbb{N}$ such that $X = \delta\mathbb{N} \setminus \mathbb{N}$?

A few more examples

A compact space X is a soft remainder of $\mathbb{N}$ if

1. X is a remainder and $\chi(x, X) < \mathfrak{p}$ for all $x \in X$
2. $w(X) < \mathfrak{p}$ — a special case of 1
3. X is perfectly normal — also a special case of 1.

In all cases: every compactification with X as a remainder is soft.
Because there are, for every point in X, plenty of sequences in $\mathbb{N}$ that converge to that point.
And we can repurpose the proof for metric compactifications.

An answer

The Continuum Hypothesis implies "Yes".

Theorem

The Continuum Hypothesis implies that every compact Hausdorff space of weight at most $\mathfrak{c}$ is the remainder in some soft compactification of $\mathbb{N}$.

Parovichenko's theorem says: the Continuum Hypothesis implies that X is the remainder in some compactification of $\mathbb{N}$ if and only if X is compact Hausdorff and of weight at most $\mathfrak{c}$.

About the proof

Parovichenko's proof has two ingredients.

Every compact Hausdorff space of weight at most $\aleph_1$ is a remainder in some compactification of $\mathbb{N}$.

Every remainder has weight at most $\mathfrak{c}$.

The Continuum Hypothesis combines the two into a characterization.

This will not work in this case, as we shall see anon.

About the proof

We assume CH and build, given a candidate space X, a soft compactification of $\mathbb{N}$ with X as its remainder.

By making sure we can repurpose the proof for metric compactifications again.

About the proof

Embed X in the Tychonoff cube $[0,1]^{\aleph_1}$.

Recursively find $f_\alpha : \mathbb{N} \to [0,1]$ such that, with f the diagonal map, $\operatorname{cl} f[\mathbb{N}] = f[\mathbb{N}] \cup X$ is a compactification of X.

Along the way construct an almost disjoint family $\mathcal{S}$ on $\mathbb{N}$ such that for every $S \in \mathcal{S}$ the image $f[S]$ converges to a point, x_S, of X.

This we can do without CH.

About the proof

We need CH for: if $\operatorname{cl} f[A]$ and $\operatorname{cl} f[B]$ intersect then there are S and T in $\mathcal{S}$ such that $S \cap A$ and $T \cap B$ are infinite and $x_S = x_T$.

Then we can repurpose the metric proof:
interchanging S and T will give an autohomeomorphism as required.

$\omega_1 + 1$

Here is an easy space, the ordinal space $\omega_1 + 1$.

Using a tower $\langle T_\alpha : \alpha \in \omega_1 \rangle$ it is easy to construct a compactification of $\mathbb{N}$ with $\omega_1 + 1$ as its remainder.

And conversely, if we have such a compactification choose disjoint open L_α and U_α, with $[0, \alpha] \subseteq L_\alpha$ and $[\alpha + 1, \omega_1] \subseteq U_\alpha$.
Then setting $T_\alpha = \mathbb{N} \cap L_\alpha$ gives us a tower.

$\omega_1 + 1$

"Every compactification of $\mathbb{N}$ with $\omega_1 + 1$ as its remainder is soft" is equivalent to
$\mathfrak{t} > \aleph_1$

If $\mathfrak{t} = \aleph_1$ take a tower with $\sup_\alpha T_\alpha = \mathbb{N}$ (mod finite) and make the corresponding compactification $\tau\mathbb{N}$.

Exercise: show that $\tau\mathbb{N}$ is soft. (Hint: $\operatorname{cl} A \cap \operatorname{cl} B \neq \{\omega_1\}$.)

Take the one-point compactification $\alpha\mathbb{N}$ and in the sum $\tau\mathbb{N} \oplus \alpha\mathbb{N}$ identify ω_1 and ∞ to one point.

Exercise: show that this compactification (of the union of the two copies of $\mathbb{N}$) is *not* soft.

$\omega_1 + 1$

"Every compactification of $\mathbb{N}$ with $\omega_1 + 1$ as its remainder is soft"
is equivalent to
$\mathfrak{t} > \aleph_1$

If $\mathfrak{t} > \aleph_1$ and we take any compactification $\tau\mathbb{N}$ from a tower then
$\operatorname{cl} A \cap \operatorname{cl} B = \{\omega_1\}$ is possible but now,
because $\mathfrak{t} > \aleph_1$,
A and B contain sequences that converge to ω_1.

$\omega_1 + 1 + \omega_1^\star$

Take two copies of $\omega_1 + 1$ and identify the two copies of the point ω_1.

Using a principle devised by Alan: it is consistent that there is no soft compactification of $\mathbb{N}$ with this space as its remainder.

Very roughly: every compactification with $\omega_1 + 1 + \omega_1^\star$ as its remainder looks like the sum of two compactifications from maximal ω_1-towers identified at the end points.

Here is where we see the need for CH:
Parovichenko's first ingredient is not available separately.

Cubes

The Cantor cube 2^{ω_1} and the Tychonoff cube $[0,1]^{\omega_1}$ are soft remainders.

Clear if $\mathfrak{t} > \aleph_1$

A fair amount of work if $\mathfrak{t} = \aleph_1$

But we use convergent sequences again
and the maximal tower is very instrumental in ensuring we have enough of them.

Questions

What about separable compact spaces?

In particular $2^{\mathfrak{c}}$ and $[0,1]^{\mathfrak{c}}$?

In particular $2^{\mathfrak{t}}$ and $[0,1]^{\mathfrak{t}}$?

In the original post there is also:
Is a remainder that is Fréchet-Urysohn also a soft remainder?

Light reading

Website: `fa.ewi.tudelft.nl/~hart`

Taras Banakh and Igor Protasov,
Constructing a coarse space with a given Higson or binary corona, Topology and its Applications **284** (2020) 107366, 20

Alan Dow and Klaas Pieter Hart,
All Parovichenko spaces may be soft-Parovichenko,
`https://arxiv.org/abs/1811.03912`.