A zero-dimensional F-space that is not strongly zero-dimensional

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The problem

Is there a zero-dimensional F-space that is not strongly zero-dimensional?

The terms:

Zero-dimensional the clopen sets form a base (plus T_1)

Strongly zero-dimensional if two sets can be separated by a continuous function to $\mathbb{R}$ then they can be separated by a continuous function to $\{-1, 1\}$.
A bit more about that
For normal spaces: disjoint closed sets can be separated by clopen sets.
For Tychonoff spaces: the Čech-Stone compactification is zero-dimensional.

The problem

F-space if $f : X \to \mathbb{R}$ is continuous then there is (another) continuous function $k : X \to \mathbb{R}$ such that $f = k \cdot |f|$.

That almost looks like strong zero-dimensionality (picture).

That picture was misleading; there are (compact) connected F-spaces (better picture).

Implications

Zero-dimensional implies strongly zero-dimensional for

Compact spaces: just like "regular implies normal"

Lindelöf spaces: same reason (Lemma 1.5.15 in Engelking's book)

Hence in particular: separable metrizable spaces

Examples

Dowker's example: a subspace M of $\omega_1 \times [0,1]$ that is normal and zero-dimensional, but not strongly zero-dimensional. (More about this example later.)

Prabir Roy's metrizable space that is zero-dimensional but not strongly so.
So separable is really necessary. (Really: John Kulesza has an example of weight $\aleph_1$.)

The question

Why only now?

The question for F-spaces must have been around long but we haven't found any explicit statement before a few years ago (2016-05-20) on MathOverFlow.
`https://mathoverflow.net/questions/239324`

With this comment: "If I remember correctly, I have at a conference heard Alan Dow refer to this problem as an open problem."

Dowker's example M

As this was the second-order inspiration for our example we'll look at this one first.

Take $\aleph_1$ many cosets of $\mathbb{Q}$ in $\mathbb{R}$, say $\langle Q_\alpha : \alpha \in \omega_1 \rangle$. (But not $\mathbb{Q}$ itself.)

We abbreviate $\bigcup_{\beta \geqslant \alpha} Q_\beta$ as T_α.

Define

$$M = \left\{ \langle \alpha, x \rangle : \alpha \in \omega_1 \text{ and } x \notin T_\alpha \right\} \subseteq (\omega_1 + 1) \times [0, 1]$$

so the set of xs with $\langle \alpha, x \rangle \in M$ grows with α.

Dowker's example M

Properties of M:

zero-dimensional for $\langle \alpha, x \rangle$ use vertical intervals with end points in Q_α

normal Pressing Down Lemma

not strongly zero-dimensional M is C^*-embedded in $M \cup \big(\{\omega_1\} \times [0,1]\big)$

Dowker's example M modified

We keep the notation but use quotients, not subspaces.

Let $\mathbb{A}$ be Alexandroff's split interval; that is,

$$\mathbb{A} = \{\langle x, i\rangle \in [0,1] \times 2 : (x = 0 \to i = 1) \wedge (x = 1 \to i = 0)\}$$

ordered lexicographically (with order topology).

Take the following quotient of $(\omega_1 + 1) \times \mathbb{A}$:

$$N^+ = \Big\{\big\langle \alpha, \langle x, i\rangle\big\rangle : \text{if } x \notin T_\alpha \text{ then } i = 0\Big\}$$

meaning: identify $\big\langle \alpha, \langle x, 0\rangle\big\rangle$ and $\big\langle \alpha, \langle x, 1\rangle\big\rangle$ whenever $x \notin T_\alpha$.

$T_{\omega_1} = \emptyset$, so at ω_1 we have $[0,1]$.

Dowker's example M modified

So: more and more neighbours are identified as we go out to ω_1.

At ω_1 we identify all neighbours and get $[0, 1]$.

We let $N = N^+ \setminus \bigl(\{\omega_1\} \times [0, 1]\bigr)$.

Properties of N:

zero-dimensional for $\langle \alpha, \langle x, i \rangle \rangle$ use vertical intervals with end points in Q_α

normal Pressing Down Lemma (or: the quotient map is closed)

not strongly zero-dimensional N is C^*-embedded in N^+

locally compact clear; this was the reason for the modification

Our example

We start with an ordered continuum K with a dense subset D that is enumerated as $\langle d_\alpha : \alpha \in \omega_2 \rangle$ in such a way that every tail $T_\alpha = \{d_\beta : \beta \geqslant \alpha\}$ is dense in K.

If you like $\neg$CH do like Dowker: $K = [0,1]$ and take $\aleph_2$ many cosets of $\mathbb{Q}$ $(Q_\alpha \cap (0,1) = \{d_{\omega\alpha+n} : n \in \omega\})$.

Our example

If you like ZFC better let M be the linearly ordered sum $\omega_2^\star + \{0\} + \omega_2$, where $\omega_2^\star$ denotes ω_2 with its order reversed.

Following Hausdorff (1906) we let $L = M_0(\omega)$, that is, the set $\{x \in M^\omega : \{m : x_n \neq 0\}$ is finite$\}$, ordered lexicographically. It is elementary to verify that the linear order is dense, in fact every interval has cardinality $\aleph_2$, and has no smallest or largest element.

Let K be the Dedekind completion of L; then L itself is the required dense set.

Our example

We let

$$K_\alpha = \{\langle x, i\rangle \in K \times 2 : \text{if } x \notin T_\alpha \text{ then } i = 0\}$$

The larger α the fewer points are split, and $K_{\omega_2} = K$ (and $T_{\omega_2} = \emptyset$).

We take a quotient of $(\omega_2 + 1) \times K_0$, as above:

$$N^+ = \Big\{\big\langle \alpha, \langle x, i\rangle\big\rangle : \text{if } x \notin T_\alpha \text{ then } i = 0\Big\}$$

Then $N = N^+ \setminus \big(\{\omega_2\} \times K\big)$ is just like our modification of Dowker's M.

Except that it is not an F-space.

Our example

First: $\omega_2 + 1$ has too many convergent sequences; we replace it by its G_δ-modification $(\omega_2 + 1)_\delta$.

Second: ordered compacta have too many convergent sequences; we replace them by Čech-Stone remainders.

Our example

Our starting point is $(\omega_2 + 1)_\delta \times \beta(\omega \times K_0)$.

We need some maps for administrative purposes:

- $q_{\beta,\alpha} : K_\beta \to K_\alpha$, where $\beta < \alpha$, is the natural map that identifies $\langle d_\gamma, 0\rangle$ and $\langle d_\gamma, 1\rangle$ when $\beta \leqslant \gamma < \alpha$;
- q_α abbreviates $q_{0,\alpha}$.

Our example

We have the maps $Q_\alpha : \beta(\omega \times K_0) \to \beta(\omega \times K_\alpha)$ induced by the maps q_α.

These induce a map Q from $(\omega_2 + 1)_\delta \times \beta(\omega \times K_0)$ onto

$$Y = \bigcup_{\alpha \leqslant \omega_2} \{\alpha\} \times \beta(\omega \times K_\alpha)$$

We give Y the quotient topology that it gets from the product and Q.
Fairly elementary: Q is a closed map.

Alas, Y is not an F-space, because it contains copies of the K_α.

Our example

For every α we let $X_\alpha = (\omega \times K_\alpha)^*$ (Čech-Stone remainder of course).

Our space is

$$X = \bigcup_{\alpha \in \omega_2} \{\alpha\} \times X_\alpha$$

and we let $X^+ = X \cup \left(\{\omega_2\} \times X_{\omega_2}\right)$, both as subspaces of the quotient of course.

Our example

Properties of X:

zero-dimensional for $\langle\alpha, x\rangle$ use vertical intervals with end points in T_α to generate the necessary clopen sets

not strongly zero-dimensional X is C^*-embedded in X^+ and X_{ω_2} is one-dimensional

F-space given $f : X^+ \to \mathbb{R}$ there is for every α of uncountable cofinality a $\beta < \alpha$ such that $f \circ Q$ is constant on all sets of the form $(\beta, \alpha] \times \{x\}$
Use that X_α is an F-space to find $k : X_\alpha \to \mathbb{R}$ such that $f = k \cdot |f|$ on $\{\alpha\} \times X_\alpha$
Extend k to $\bigcup_{\beta<\gamma<\alpha}\{\gamma\} \times X_\gamma$ by $k(\gamma, x) = k\big(\alpha, Q_{\gamma,\alpha}(x)\big)$

Variations

Take $n \in \mathbb{N}$ and replace K or K_α by K^n or K_α^n everywhere.
Then X is zero-dimensional, but $\dim X = n$.

Or take K^ω or K_α^ω everywhere: we get $\dim X = \infty$.

We can even get X_{ω_2} to be indecomposable and under $\neg$CH equal to $[0, \infty)^*$.