Universal autohomeomorphisms of $\mathbb{N}^*$

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Košice, 4 april 2022: 14:00 – 15:00

What are we talking about?

There are many types of universal objects.

In Topology we have two notions:

A member, X, of a class $\mathcal{C}$ is *universal* is every member of $\mathcal{C}$ can be embedded into X.

A member, X, of a class $\mathcal{C}$ is *universal* is every member of $\mathcal{C}$ is a continuous image of X.

Embedding

Classical examples:

$[0,1]^\kappa$ is universal for the class of all compact Hausdorff spaces (even all Tychonoff spaces) of weight at most κ

$\{0,1\}^\kappa$ is universal for the class of all zero-dimensional compact Hausdorff spaces (even all zero-dimensional spaces) of weight at most κ

Continuous image

This seems to be more difficult:

The Cantor set is a universal compact metrizable space.

The space $\mathbb{N}^*$ is a universal compact space of weight $\mathfrak{c}$,
if the Continuum Hypothesis holds.

It is consistent that there is no universal compact space of weight $\mathfrak{c}$.

Universal autohomeomorphisms (embedding)

A pair (X, h), where X is a space and $h : X \to X$ is an autohomeomorphism, is universal for a class of similar pairs if for every such pair (Y, g) there is an embedding $e : Y \to X$ such that $h \circ e = e \circ g$.

A pair (X, h), where X is a space and $h : X \to X$ is an autohomeomorphism, is universal for a class of similar pairs if for every such pair (Y, g) there is a continuous surjection $s : X \to Y$ such that $g \circ s = s \circ h$.

We leave the second type of universality for another time.

Universal autohomeomorphisms

A large source of such pairs is obtained as follows:

Take a space X such that X is homeomorphic to X^ω.

Let $h : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ be the shift map: $h(x)_n = x_{n+1}$ (shift to the left).

If $Y \subseteq X$ and $g : Y \to Y$ is an autohomeomorphism then define $e : Y \to X^{\mathbb{Z}}$ by
$e(y) = \langle g^n(y) : n \in \mathbb{Z} \rangle$.
Now note: $h(e(y)) = e(g(y))$ for all y.

Also note that Y is re-embedded into X.

Universal autohomeomorphisms

So, we have universal autohomeomorphisms for all pairs (Y, g) where

- Y is compact Hausdorff (even Tychonoff) of weight at most κ
- Y is zero-dimensional compact Hausdorff (even Tychonoff) of weight at most κ

Our goal

An autohomeomorphism h of $\mathbb{N}^*$ that is universal for all pairs (Y, g), where Y is a *closed* subspace of $\mathbb{N}^*$.

Is that possible?
What would it look like?

Unfortunately $\mathbb{N}^*$ is not homeomorphic to its countable power, so we need a new idea.

It cannot be too easy

Remember: $\beta\mathbb{N}$ is the set of ultrafilters on $\mathbb{N}$, with the family

$$\{\overline{A} : A \subseteq \mathbb{N}\}$$

as a base, where $\overline{A} = \{u : A \in u\}$.

The sets $\overline{A}$ are closed and open: $\overline{\mathbb{N} \setminus A} = \beta\mathbb{N} \setminus \overline{A}$.

Also: $\overline{A}$ is the closure of A if $A \subseteq \mathbb{N}$.

And $A^* = \overline{A} \setminus A$, and so $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$.

It cannot be too easy

If $\pi : \mathbb{N} \to \mathbb{N}$ is a bijection then it induces an autohomeomorphism of $\beta\mathbb{N}$, and also an autohomeomorphism of $\mathbb{N}^*$.

More generally: if A and B are co-finite and $\pi : A \to B$ is a bijection then it determines a homeomorphism $\pi^* : A^* \to B^*$.
But $A^* = B^* = \mathbb{N}^*$, so π^* is in fact an autohomeomorphism of $\mathbb{N}^*$.

These are the easy ones; they are called the trivial autohomeomorphisms of $\mathbb{N}^*$.

It cannot be too easy

Theorem 1

No trivial autohomeomorphism is universal.

An (easy) exercise: is π^* is a trivial autohomeomorphism then $\{u : \pi^*(u) = u\}$ is clopen (Hint: it is $\{n : \pi(n) = n\}^*$).

We find a closed subset Y of $\mathbb{N}^*$ with an autohomeomorphism g with a single non-isolated fixed point.

Such a pair cannot be embedded into $(\mathbb{N}^*, \pi^*)$. Picture

It cannot be too easy

Let $X = \omega_1 + 1$ with the topology where each $\alpha \in \omega_1$ is isolated and $\{(\alpha, \omega_1] : \alpha < \omega_1\}$ is a local base at ω_1.

Let $Y = \beta X$ (the Čech-Stone compactification).

Define $g : Y \to Y$ by

- $g(\omega_1) = \omega_1$
- $g(2\alpha) = 2\alpha + 1$
- $g(2\alpha + 1) = 2\alpha$

and let Čech and Stone extend the map to the rest of Y.

Then ω_1 is the only fixed point of g.

It cannot be too easy

And now we cheat:

Proposition [Eric van Douwen]

The space Y can be embedded into $\mathbb{N}^*$.

So, no trivial autohomeomorphism is universal.

And: in models where all autohomeomorphisms of $\mathbb{N}^*$ are trivial (Shelah, PFA, MA+OCA, ...) there are no universal autohomeomorphisms of $\mathbb{N}^*$.

On the other hand

Let us assume the Continuum Hypothesis. (Let us look at the friendly head of $\mathbb{N}^*$.)

Many things go 'right' if we assume CH.

And we shall use a few of those 'correct' consequences to prove

Theorem 2
CH implies that there is a universal autohomeomorphism of $\mathbb{N}^*$.

The construction

How to prove Theorem 2?

In two big steps:

Build a space Z with a homeomorphism h that is universal for all pairs (Y, g) with Y closed in $\mathbb{N}^*$ and g an autohomeomorphism of Y.

Embed Z into $\mathbb{N}^*$ in such a way that there is an autohomeomorphism H of $\mathbb{N}^*$ that extends (the copy of) h.

The first step

We let Aut denote the autohomeomorphism group of $\mathbb{N}^*$.

It carries a natural topology: the compact-open topology.
This makes it a topological group.

Then the natural action $\sigma : \mathrm{Aut} \times \mathbb{N}^* \to \mathbb{N}^*$, defined by by $\sigma(f, u) = f(u)$ is continuous.

Then $h : \mathrm{Aut} \times \mathbb{N}^* \to \mathrm{Aut} \times \mathbb{N}^*$, defined by $h(f, u) = (f, f(u))$, is continuous (both coordinate maps are continuous) and bijective.

The inverse of h is given by $h^{-1}(f, u) = (f, f^{-1}(u))$ is continuous too.
First coordinate: certainly.
Second coordinate: $(f, u) \mapsto (f^{-1}, u) \mapsto \sigma(f^{-1}, u)$.

But ...

The first step

... this is not quite our space Z yet.

It is universal for the collection of pairs that we specified.

Indeed, let (Y, g) be given.

Correct result 1: Y can be (re)embedded into $\mathbb{N}^*$ so that it becomes a closed P-set.
Correct result 2: g can be extended to an autohomeomorphism g^+ of $\mathbb{N}^*$
(van Douwen and van Mill)

Look at the (re)embedded copy of Y in $\{g^+\} \times \mathbb{N}^*$:
since h acts like g^+ on that 'vertical line' it extends that copy of g.

The second step

We change the topology on Aut a bit to get a space that can be embedded into $\mathbb{N}^*$.

We had the compact-open topology τ; we take its G_δ-modification τ_δ (generated by the G_δ-sets).

Then $(\mathrm{Aut}, \tau_\delta)$ is also a topological group.

Our space Z is the product $\mathrm{Aut} \times \mathbb{N}^*$, where Aut carries τ_δ.

The map h is also an autohomeomorphism of Z and everything we established for the normal product holds for Z too.

The pair (Z, h) is universal for all pairs (Y, g), because the topology on the vertical lines does not change.

The second step

Also, Z is a strongly zero-dimensional F-space.

In fact, every open cover has a pairwise disjoint open refinement.

Here is where the topology τ_δ is used.
Negrepontis: the product of a P-space and an F-space is again an F-space.

An F-space is (can be) defined by: if $f : X \to \mathbb{R}$ is continuous then there is another continuous function $k : X \to \mathbb{R}$ such that $f = k \cdot |f|$.

The second step

Now: Z is a strongly zero-dimensional F-space of weight $\mathfrak{c}$.

It has a compactification K that is also a zero-dimensional F-space of weight $\mathfrak{c}$ and such that h has an extension to an autohomeomorphism h^* of K.

Note K is not βZ, but the result of an application of the Löwenheim-Skolem theorem to the Boolean algebra of clopen sets.

The second step

We apply Correct result 1 again: K can be embedded into $\mathbb{N}^*$ as a P-set.

And by Correct result 2 again: h^* has an extension to an autohomeomorphism of $\mathbb{N}^*$.

Done!

Further reading

`https://fa.ewi.tudelft.nl/~hart`

K. P. Hart and Jan van Mill, *Universal autohomeomorphisms of* $\mathbb{N}^*$, to appear in Proceedings of the American Mathematical Society, Series B.

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