150 Years of Uncountability of $\mathbb{R}$

Tá scéilín agam

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Cyberspace, 1 December 2023: 10:00–11:00

The question


Georg Cantor (1845–1918)

Richard Dedekind (1831–1916)

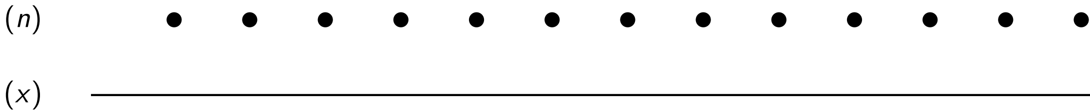

Halle, d. 29ten Nov. 73.

Man nehme den Inbegriff aller positiven ganzzahligen Individuen n und bezeichne ihn mit (n); ferner denke man sich etwa den Inbegriff aller positiven reellen Zahlgrössen x und bezeichne ihn mit (x); so ist die Frage einfach die, ob sich (n) dem (x) so zuordenen lasse, dass zu jedem Individuum des einen Inbegriffes ein und nur eines des andern gehört?

That sounds way better than: "Is there a bijection between $\mathbb{N}$ and $(0, \infty)$?"
(Peano's definition of function of from 1911.)

Remarks by Cantor

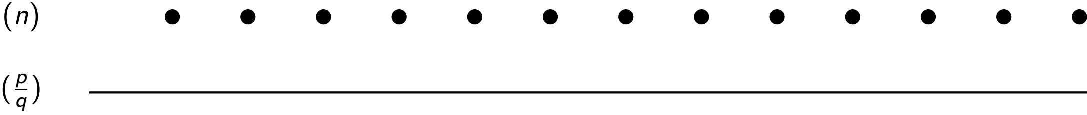
At first sight one would say that it is not possible because (n) consists of discrete parts and (x) is a continuum.



But that says nothing, and even though I think it is impossible, I cannot find the true reason, and that is what I am looking for; maybe that reason is actually quite simple.

Remarks by Cantor

Would one not at first sight also tend to think that the (n) and the whole $(\frac{p}{q})$ of the positive rational numbers will not let themselves be coupled in a one-to-one fashion?



But it is however not difficult to show that the latter is possible and that (n) can be coupled in a one-to-one fashion to $(a_{n_1,n_2,\ldots,n_\nu})$, where $n_1, n_2, \ldots, n_\nu$ are unlimited positive integer indices of an arbitrary quantity ν.

Apparently $(a_{n_1,n_2,\ldots,n_\nu})$ instead of the ordered ν-tuple $(n_1, n_2, \ldots, n_\nu)$

Next letter: Halle d. 2ten December 73.

Apparently Dedekind had written back that he did not know either.
Cantor showed himself put at ease: it was not him but the question.

Also: I never occupied myself seriously with this because it has no practical use for me, and I agree with you that the problem does not deserve that much effort.

But still: it would be nice if it was resolved, because if the answer would turn out to be **no** then we would have a new proof of Liouville's theorem that there are transcendental numbers.

Something was afoot here . . .

Next letter: Halle d. 2^{ten} December 73.

... namely

Your proof that (n) will let itself be coupled univocally to the field of algebraic numbers is roughly the same as my proof for $(a_{n_1,n_2,\ldots,n_\nu})$: write $n_1^2 + n_2^2 + \cdots + n_\nu^2 = \mathfrak{N}$ and order the elements after that quantity.

Next letter: Halle d. 2ten December 73.

For the positive rational numbers: for every n we have finitely many positive fractions $\frac{p}{q}$ with $p^2 + q^2 = n$. Sort the fractions first by the group that they are in and inside each group by their numerators.

So:

$$\underbrace{\frac{1}{1}}_{2}, \underbrace{\frac{1}{2}, \frac{2}{1}}_{5}, \underbrace{\frac{2}{2}}_{8}, \underbrace{\frac{1}{3}, \frac{3}{1}}_{10}, \underbrace{\frac{2}{3}, \frac{3}{2}}_{13}, \underbrace{\frac{1}{4}, \frac{4}{1}}_{17}, \dots$$

Using $p + q = n$ makes it looks somewhat nicer:

$$\underbrace{\frac{1}{1}}_{2}, \underbrace{\frac{1}{2}, \frac{2}{1}}_{3}, \underbrace{\frac{1}{3}, \frac{2}{2}, \frac{3}{1}}_{4}, \underbrace{\frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}}_{5}, \dots$$

Next letter: Halle d. 2^{ten} December 73.

Is it not nice that one, as you showed, can speak of the *n*th algebraic number?

And as you remarked correctly we can reformulate the problem as: can (n) be coupled univocally to the entity $(a_{n_1,n_2,\ldots})$ where the n_1, n_2, ... are unlimited positive whole numbers and infinite in number.

$\Big[$ The set of sequences of natural numbers in disguise. $\Big]$

Dedekind would later use the countability of the field of algebraic numbers in *Über die Permutationen des Körpers aller algebraischen Zahlen* (1901).

The answer

Halle d. 7ten December 73.

In den letzten Tagen habe ich die Zeit gehabt, etwas nachhaltiger meine Ihnen gegenüber ausgesprochene Vermuthung zu verfolgen; erste heute glaube ich mit der Sache fertig geworden zu sein; sollte ich mich jedoch täuschen, so finde ich gewiss keinen nachsichtigeren Beurtheiler, als Sie. Ich nehme mir also die Freiheit, Ihrem Urtheile zu unterbreiten, was soeben in der Unvollkommenheit des ersten Conceptes zu Papier gebracht is.

Then there is a proof.

And then

So glaube ich schliesslich zum Grunde gekommen zu sein, weshalb sich der in meinen früheren Briefe mit (x) bezeichnete Inbegriff **nicht** dem mit (n) bezeichneten eindeutig zuordnen lässt.

Triple celebration?

So, this coming Tuesday we may very well be celebrating three things:

- St. Nicholas day
- Cantor getting the first inklings of a proof
- Alan's birthday

Commercial!!!!

You can read the proof in Pythagoras.

See the issue of April 2018

in Dutch of course.

The proof from the letter

Assume we can list all positive numbers $\omega < 1$ in a sequence:

$$\omega_1, \omega_2, \omega_3, \ldots, \omega_n, \ldots \qquad (*)$$

After ω_1 we take the first term ω_α larger than ω_1, and next the first term ω_β larger than ω_α, and so on.
Write $\omega_1 = \omega_1^1$, $\omega_\alpha = \omega_1^2$, $\omega_\beta = \omega_1^3$, and so on; in this way we extract from $(*)$ an infinite sequence:

$$\omega_1^1, \omega_1^2, \omega_1^3, \ldots, \omega_1^n, \ldots$$

The first term from the remaining sequence we call ω_2^1, the first term that is larger we call ω_2^2, and so on; we extirpate a second subsequence:

$$\omega_2^1, \omega_2^2, \omega_2^3, \ldots, \omega_2^n, \ldots$$

. . . .

The proof from the letter

We keep going and divide our sequence $(*)$ in infinitely many subsequences:

$$\omega_1^1, \omega_1^2, \omega_1^3, \ldots, \omega_1^n, \ldots \tag{1}$$
$$\omega_2^1, \omega_2^2, \omega_2^3, \ldots, \omega_2^n, \ldots \tag{2}$$
$$\omega_3^1, \omega_3^2, \omega_3^3, \ldots, \omega_3^n, \ldots \tag{3}$$

where we always have: $\omega_k^\lambda < \omega_k^{\lambda+1}$

The proof from the letter

Take an interval $(p \cdots q)$ that contains no term of the sequence (1), say inside $(\omega_1^1 \ldots \omega_1^2)$.

The terms of the second, and also the third, sequence may all lie outside $(p \cdots q)$, but there must be a (first) sequence, say the kth one, of which not all terms lie outside $(p \cdots q)$ (otherwise no term of the whole sequence lies inside that interval and that is impossible).

Then one can take an interval $(p' \cdots q')$ inside $(p \cdots q)$ that contains no term of the kth sequence, and hence also no terms from the earlier subsequences.

Then, again, there is a (first) sequence, say the k'th, of which not all terms lie outside $(p' \cdots q')$.

Then one can take a third interval $(p'' \cdots q'')$ inside $(p' \cdots q')$ that contains no term of the k'th sequence, and hence no terms of any earlier subsequence.

The proof from the letter

And we go on like this, we get a decreasing sequence of intervals

$$(p \cdots q),\ (p' \cdots q'),\ (p'' \cdots q''),\ \ldots$$

in such a way that the terms of the subsequences 1 through $k-1$ lie outside $(p \cdots q)$, and the terms of the subsequences k through $k'-1$ lie outside $(p' \cdots q')$, and the terms of the subsequences k' through $k''-1$ lie outside $(p'' \cdots q'')$, and so on.

Then there is a number, η say, that lies in all those intervals, and of which one sees at once that it lies between 0 and 1 and does not occur in any one of the subsequences, and hence not in the original sequence $(*)$.

So our initial assumption was incorrect.

Twee days later

Halle d. 9^{ten} December 73.

I have found a simpler proof, without splitting into subsequences.

I show directly that if I start with a sequence

$$\omega_1, \omega_2, \omega_3, \ldots, \omega_n, \ldots \qquad (*)$$

I can determine in any given interval $(\alpha \,..\, \beta)$ a number η that does not occur in the sequence $(*)$.

From this it follows immediately that (n) and (x) can not be coupled in a one-to-one fashion and I conclude that there are differences between these two sets that I could not fathom hitherto.

Berlin d. 25^{ten} December 73.

Cantor writes:

although did not want to publish the result, that I recently discussed with you, I was urged to do so anyway.
I had told Mr. Weierstrass about the result on the 22^{nd} and I showed him the proof on the 23^{rd}.
He thought I should publish it.
I wrote a short article with the title

Ueber eine Eigenschaft des Inbegriffes aller reelen algebraischen Zahlen

The algebraic numbers

First there is the proof (by Dedekind) that there are ‘only’ countably many algebraic numbers.
Every algebraic number, ω, is the solution of a polynomial equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

with every a_i an integer, a_0 positive and such that the greatest common divisor of the a_i is equal to 1, and such that the equation is irreducible.

In that case the equation is determined completely by ω (and hence also by the other solutions).

The number

$$N = (n - 1) + |a_0| + |a_1| + \cdots + |a_n|$$

is called the *height* of ω (and also of the equation).

The algebraic numbers

Every natural number N is the height of finitely many equations and hence of finitely many algebraic numbers, these form a set A_N.

Every A_N is ordered by the normal order of $\mathbb{R}$.

Enumerate the A_N in order: first A_1, then A_2, then A_3 and so on (each time from left to right)

This results in an enumeration of the set (ω) of all real algebraic numbers.

NB The number of polynomials of height N has been counted, see sequence A005409 in the OEIS

The theorem

Wenn eine nach irgend einem Gesetze gegebene unendliche Reihe von einander verschiedener reeller Zahlgrössen:

$$\omega_1, \omega_2, \ldots, \omega_\nu, \ldots \tag{4.}$$

vorliegt, so lässt sich in jedem vorgegebenen Intervalle $(\alpha \ldots \beta)$ eine Zahl η (und folglich unendlich viele solcher Zahlen) bestimmen, welche in der Reihe (4.) nicht vorkommt; dies soll nun bewiesen werden.

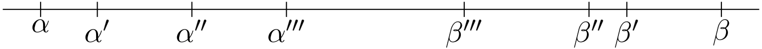
The published proof

Here is the 'simpler' proof of the non-existence of a univocal coupling between (n) and (x).
Given a sequence

$$\omega_1, \omega_2, \omega_3, \ldots, \omega_\nu, \ldots$$

of positive real numbers and an interval $(\alpha \ldots \beta)$.

The published proof



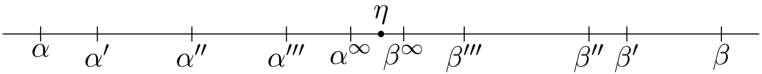
- Let α' and β' be the *first* two terms of the sequence (if any) that lie in $(\alpha \ldots \beta)$ and such that $\alpha < \alpha' < \beta' < \beta$.
- Let α'' and β'' be the *first* two terms of the sequence (if any) that lie in $(\alpha' \ldots \beta')$ and such that $\alpha' < \alpha'' < \beta'' < \beta'$.
- Let α''' and β''' be the *first* two terms of the sequence (if any) that lie in $(\alpha'' \ldots \beta'')$ and such that $\alpha'' < \alpha''' < \beta''' < \beta''$.
- and so on

The published proof

Two cases:

1. The number of intervals is finite; then we find an interval $(\alpha^{(\nu)} \dots \beta^{(\nu)})$ with at most one term of the sequence in it. Then, clearly, we are done.

The published proof



2. The number of intervals is infinite; then we find an increasing sequence

$$\alpha < \alpha' < \alpha'' < \cdots$$

that is bounded above (by β) and hence convergent, with limit α^∞; and a decreasing sequence

$$\beta > \beta' > \beta'' > \cdots$$

that is bounded below (by α) and hence convergent, with limit β^∞.
Take η in the interval $(\alpha^\infty \ldots \beta^\infty)$ (NB $\alpha^\infty = \beta^\infty$ is very well possible).

Now ponder why η does not occur in the sequence.

Review

JFM 06.0057.01

Trotzdem in der Nähe jeder beliebig gegebenen reellen Zahl unendlich viele reelle algebraische Zahlen liegen, kann man dennoch den Inbegriff aller reellen algebraischen Zahlen dem aller positiven ganzen Zahlen zuordnen, so dass jede der einen Reihe nur einer der andern entspricht. Da sich nun weiter zeigen lässt, dass wenn eine beliebige Reihe reeller Zahlengrössen vorliegt, man in jedem Intervalle Zahlen bestimmen kann, die nicht zur Reihe gehören, so folgt ein Beweis des zuerst von Liouville gegebenen Satzes, dass in jedem reellen Intervalle unendlich viele transcendente Zahlen vorhanden sind.

Reviewer: Netto, Dr. (Berlin)

A new question

Halle d. 5$^{\text{ten}}$ Januar 74.

Lässt sich eine Flache (etwa ein Quadrat mit Einschluss der Begrenzung) eindeutig auf eine Linie (etwa eine gerade Strecke mit Einschluss der Endpunkte) eindeutig beziehen, so dass zu jedem Puncte der Fläche ein Punct der Linie und umgekehrt zu jedem Puncte der Linie ein Punct der Fläche gehört?

Mir will es im Augenblick noch scheinen, dass die Beantwortung dieser Fragen, — obgleich man auch hier zum Nein sich so gedrängt sieht, dass man de Beweis dazu fast für überflüssig halte möchte, — grosse Schwierigkeiten hat.

A new question

Halle 18. Mai 74.

. . . in Berlin wurde mir von meinem Freunde, dem ich dieselbe Schwierigkeit vorlegte, die Sache gewissermassen als absurd erklärt, da es sich von selbst verstünde, dass zwei unabhänginge Veränderliche sich nicht auf eine zurückführen lassen.

An answer

Halle d. 20^{ten} Juni 1877.

A fairly long letter with a proof, by splicing decimal expansions, that every finite number of variables "mit Spielraum $\geqq 0$ und $\leqq 1$" can be reduced to a single variable with the same limits.

Dedekind's answer

This does not work because of numbers with two expansions.

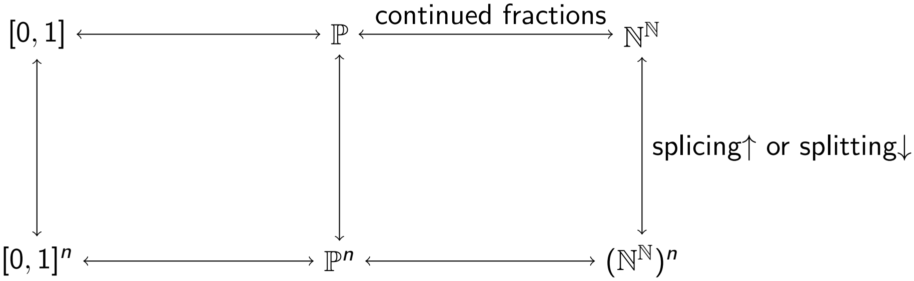
Karte : Poststempel 23.6.77.

You are right, but the objection only applies to the proof, fortunately not to the theorem. I will send a more more extensive letter in a few days.

An answer

Halle 25ten Juni 1877.

A long letter (five pages in the collected correspondence): a correct proof using continued fractions.



Consequences for dimension

Cantor: this has consequences for geometry; everybody says that you need n independent coordinates to describe n-dimensional sets and thinks that that speaks for itself. I see an error there.

Dedekind: your bijections are not continuous and I think that in (differential) geometry one should be talking about continuous maps.

Cantor: certainly, but I meant that many think that one needs n independent coordinates under all circumstances.
I agree that it is probable that the restriction to continuous maps necessitates the use of n independent coordinates. But I do not see how to prove that and it seems quite difficult to me.

Consequences for dimension

Brouwer (some thirty years later): you were right, continuous bijections leave the dimension invariant.

Light reading

Website: `fa.ewi.tudelft.nl/~hart`

Georg Cantor,
Ueber eine Eigenschaft des Inbegriffs aller reellen algebraischen Zahlen, Crelles Journal für Mathematik **77** (1874) 258–262.

Georg Cantor,
Ein Beitrag zur Mannigfaltigkeitslehre, Crelles Journal für Mathematik **84** (1877) 242–258.

K. P. Hart,
Cantors Diagonaalargument, Nieuw Archief voor Wiskunde, **16** (1) (2015), 40–43