Many subalgebras of $\mathcal{P}(\omega)/\mathit{fin}$

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A question on mathoverflow

Mad Hatter asks

Consider the algebra $B = P(\omega)/_{\mathrm{fin}}$ (the quotient of the power set of natural numbers modulo the ideal of finite sets). Is there an infinite strictly descending chain $\{A_i \mid i \in I\}$ of subalgebras of B, such that there is an embedding of A_{i+1} into A_i, but there is no embedding of A_i into A_{i+1}.

There were some comments about the possible difficulties one might encounter when proving non-embeddability of A_i into A_{i+1}.

They involved the notion of invariants: things that should 'measure' the A_i and indicate that A_i is 'too large' to fit inside A_{i+1}.

A question on mathoverflow

Most of the invariants that we know are ordinal- or cardinal-valued and decreasing sequences of these tend to be finite, so that makes it hard to create infinite decreasing chains.

However, to paraphrase a famous saying . . .

Invariants, we don't need no stinking invariants

There is another way: Mass Murder.

An old idea by Sierpiński, affectionally known as

"Sierpiński's technique of killing homeomorphisms"

allows us to line up potential bad maps and eliminate them.

Turning the question upside-down

We use Stone Duality and construct (much more than) a sequence $\langle K_n : n \in \omega \rangle$ of compact zero-dimensional spaces such that

1. K_0 is a continuous image of ω^*,
2. K_{n+1} is a continuous image of K_n (all n), and
3. K_n is not a continuous image of K_{n+1} (all n).

These spaces will all look the same superficially, with no discernible properties to distinguish them, or even prevent continuous onto maps between them.

We simply eliminate all undesirable maps.

The spaces

Consider Alexandroff's double arrow $\mathbb{A}$.

The underlying set of $\mathbb{A}$ is

$$D = \big([0,1] \times \{0,1\}\big) \setminus \big\{\langle 0,0\rangle, \langle 0,1\rangle\big\},$$

ordered lexicographically and endowed with the order topology.
We drop the points $\langle 0,0\rangle$ and $\langle 0,1\rangle$ because they would be (the only) isolated points of $\mathbb{A}$.

As $\mathbb{A}$ is separable it is a continuous image of ω^*; this take care of item 1 in our list: we can take $K_0 = \mathbb{A}$.

The spaces

There are many continuous images of $\mathbb{A}$.

For every subset X of $(0,1)$ take $\mathbb{A}_X = \{\langle x, i\rangle \in D : x \in X \to i = 0\}$, ordered lexicographically and given the order topology.

$\mathbb{A}_X$ is obtained from $\mathbb{A}$ by identifying $\langle x, 0\rangle$ and $\langle x, 1\rangle$ whenever $x \in X$.
Thus we can write, e.g., $\mathbb{A} = \mathbb{A}_\emptyset$, and $[0,1] = \mathbb{A}_{(0,1)}$.

In all our examples the complement of X will be dense in $(0,1)$ and this will ensure that $\mathbb{A}_X$ is zero-dimensional.

If $X \subseteq Y$ then there is a natural continuous surjection $s : \mathbb{A}_X \to \mathbb{A}_Y$, given by

- $s(x,i) = \langle x, i\rangle$ if $x \notin Y$;
- $s(x,i) = \langle x, 0\rangle$ if $x \in Y \setminus X$; and
- $s(x,0) = \langle x, 0\rangle$ if $x \in X$.

The spaces

We find a family $\{S_X : X \subseteq \mathfrak{c}\}$ of subsets of $(0,1)$ and put $K_X = \mathbb{A}_{S_X}$ for all X.

Whenever $X \subseteq Y$ we shall have $S_X \subseteq S_Y$ and so K_Y will be a continuous image of K_X.

All the work will go into ensuring that K_X is not a continuous image of K_Y whenever $X \not\subseteq Y$.

We get a family $\{K_X : X \subseteq \mathfrak{c}\}$ of continuous images of ω^* that is order-isomorphic to $\mathcal{P}(\mathfrak{c})$ under the relation "maps continuously onto".

By Stone Duality we get a family $\{B_X : X \subseteq \mathfrak{c}\}$ of subalgebras of $\mathcal{P}(\omega)/\mathit{fin}$ that is order-isomorphic to $\mathcal{P}(\mathfrak{c})$ under the relation "embeds into".

The sets

Some preparations before we construct the sets S_X.

Consider the set $\mathcal{F}$ of all maps f that satisfy:

- $\operatorname{dom} f$ is a co-countable subset of $[0,1]$ and
- $f : \operatorname{dom} f \to [0,1]$ is continuous.

For every $f \in \mathcal{F}$ we let

- $E(f) = \{x \in \operatorname{dom} f : f(x) = x\}$ and
- $S(f) = \{x \in \operatorname{dom} f : f(x) \neq x\}$ and

we choose a subset $C(f)$ of $\operatorname{dom} f$ such that the restriction $f : C(f) \to f\big[S(f)\big]$ is a bijection.

The sets

Useful facts:

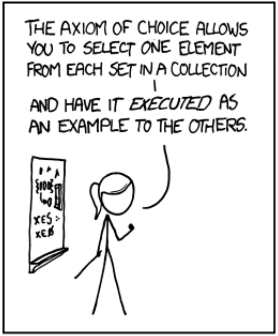
- The family $\mathcal{F}$ has cardinality $\mathfrak{c}$.
- The sets $S(f)$ and $E(f)$ are countable or of cardinality $\mathfrak{c}$.
- The set $f\big[S(f)\big]$ is countable or of cardinality $\mathfrak{c}$
- hence so is $C(f)$.

Why the last three?

$E(f)$ and $S(f)$ are Borel (G_δ even), and $f\big[S(f)\big]$ is analytic;
so if they are uncountable they even contain a copy of the Cantor set.

The sets

The members of $\mathcal{F}$ represent the potential continuous onto maps between our compacta, so they will be lined up and dealt with ...



MY MATH TEACHER WAS A BIG BELIEVER IN PROOF BY INTIMIDATION.

The sets

Proposition

There is a pairwise disjoint family $\{V\} \cup \{A_\alpha : \alpha \in \mathfrak{c}\}$ of Bernstein sets in $(0,1)$ with the following properties.
1) All are disjoint from $\mathbb{Q}$, and
2) for every $f \in \mathcal{F}$: if $f\big[S(f)\big]$, and hence $C(f)$, has cardinality $\mathfrak{c}$ then for all α the intersections $C(f) \cap A_\alpha$ and $f\big[C(f) \cap A_\alpha\big] \cap V$ both have cardinality $\mathfrak{c}$.

Bernstein set: intersects every uncountable closed subset of $[0,1]$.

Once this is done we will let, for every $X \subseteq \mathfrak{c}$:

$$S_X = \mathbb{Q} \cup \bigcup_{\alpha \in X} A_\alpha$$

The Mass Murder

Uhm, I mean: The Construction.

Enumerate the set of uncountable closed subsets of $[0, 1]$ as $\langle G_\beta : \beta \in \mathfrak{c} \rangle$,
and the set of members f of $\mathcal{F}$ for which $f\left[S(f)\right]$ has cardinality $\mathfrak{c}$ as $\langle f_\beta : \beta \in \mathfrak{c} \rangle$.
We assume each term of the sequences occurs $\mathfrak{c}$ often.

The, uhm, Construction

Well-order $\mathfrak{c}^2$ in order-type $\mathfrak{c}$, via $\prec$, and recursively choose points $a_{\alpha,\beta}$, $b_{\alpha,\beta}$, $u_{\alpha,\beta}$, and $v_{\alpha,\beta}$, as follows.

At stage $\langle\alpha,\beta\rangle$ collect $\mathbb{Q}$ and all previously chosen points $a_{\gamma,\delta}$, $b_{\gamma,\delta}$, $u_{\gamma,\delta}$, and $v_{\gamma,\delta}$, with $\langle\gamma,\delta\rangle \prec \langle\alpha,\beta\rangle$, in a set P.
Then $|P| < \mathfrak{c}$ of course.

Take $a_{\alpha,\beta}$ in $C(f_\beta) \setminus P$ and let $b_{\alpha,\beta} = f_\beta(a_{\alpha,\beta})$.
And then distinct $u_{\alpha,\beta}$ and $v_{\alpha,\beta}$ in $G_\beta \setminus \bigl(P \cup \{a_{\alpha,\beta}, b_{\alpha,\beta}\}\bigr)$

In the end let

- $A_\alpha = \{a_{\alpha,\beta} : \beta \in \mathfrak{c}\} \cup \{u_{\alpha,\beta} : \beta \in \mathfrak{c}\}$ for all α, and
- $V = \{b_{\alpha,\beta} : \langle\alpha,\beta\rangle \in \mathfrak{c}^2\} \cup \{v_{\alpha,\beta} : \langle\alpha,\beta\rangle \in \mathfrak{c}^2\}$

The Construction

Does this prove the proposition?

- We took all points outside $\mathbb{Q}$, so that is condition 1)
- Given an f and an α we have $C(f) \cap A_\alpha \supseteq \{a_{\alpha,\beta} : f = f_\beta\}$; that has cardinality $\mathfrak{c}$.
- Same f, same α: $f\left[C(f) \cap A_\alpha\right] \cap V \supseteq \{b_{\alpha,\beta} : f = f_\beta\}$, cardinality $\mathfrak{c}$ again.
- The points $u_{\alpha,\beta}$ ensure that $A_\alpha \cap G_\beta \neq \emptyset$, so A_α is a Bernstein set.
- The points $v_{\alpha,\beta}$ ensure that $V \cap G_\beta \neq \emptyset$, so V is a Bernstein set.

Verification

Remember: $S_X = \mathbb{Q} \cup \bigcup_{\alpha \in X} A_\alpha$.

For all X we have $\mathbb{Q} \subseteq S_X$ and $S_X \cap V = \emptyset$.

The former is a technical convenience, the latter shows that K_X is zero-dimensional.

We do have $S_X \subseteq S_Y$ whenever $X \subseteq Y$, so K_X does indeed map onto K_Y in that case.

If $X \not\subseteq Y$ then there is an α in $X \setminus Y$, and then $A_\alpha \subseteq S_X \setminus S_Y$.

We shall show: if $f : K_X \to K_Y$ is continuous then $f[K_X]$ is countable.

Verification

To minimize on notational complexity we formulate this as follows.

Lemma

Let X and Y be subsets of $(0,1)$ such that $\mathbb{Q} \subseteq X$ and such that there is an α for which $A_\alpha \subseteq X$ and $Y \cap (A_\alpha \cup V) = \emptyset$. Then every continuous map $s : \mathbb{A}_X \to \mathbb{A}_Y$ has a countable range.

The actual situation would be $A_\alpha \subseteq S_X$ and $S_Y \cap (A_\alpha \cup V) = \emptyset$.
We dropped the S-es for notational convenience.

Verification

Proof of the Lemma:
Because $\mathbb{Q} \subseteq X$ the rationals are not split in $\mathbb{A}_X$, we can talk unambiguously about rational intervals in $\mathbb{A}_X$.

Let $t : \mathbb{A}_Y \to [0,1]$ be the natural map, then $t \circ s : \mathbb{A}_X \to [0,1]$ is continuous.
And: because the points of X are not split in $\mathbb{A}_X$ the topology that X inherits from $\mathbb{A}_X$ is the same as the subspace topology from $[0,1]$.
And so we get a continuous map $g : X \to [0,1]$: the restriction of $t \circ s$.

Apply Lavrentieff's theorem to obtain a G_δ-set U in $[0,1]$ that contains X and a continuous extension $f : U \to [0,1]$ of g.
As A_α is a Bernstein set the complement of U in $[0,1]$ is countable, and so $f \in \mathcal{F}$.

Verification

In order to show that the range of s is countable we look at the relationship between f and s.

The continuous maps f and $t \circ s$ coincide on A_α and A_α is dense in $[0, 1]$, so the maps coincide on X.

If $x \in X$ then $f(x) = t(s(x))$ and so $s(x) = \langle f(x), 0\rangle$ or $s(x) = \langle f(x), 1\rangle$

If $x \in U \setminus X$ then $t(s(x, 0)) = f(x) = t(s(x, 1))$ by left- and right-continuity of f at x.

And so: $\bigl\{s(x, 0), s(x, 1)\bigr\} \subseteq \bigl\{\langle f(x), 0\rangle, \langle f(x), 1\rangle\bigr\}$.

We see that the range of s is contained in the union of the two sets $\bigl\{s(x, i) : x \notin U, i \in \{0, 1\}\bigr\}$ and $f[U] \times \{0, 1\}$, the first of which is countable.

So, . . . , we should to show that $f[U]$ is countable.

Verification

Step 1: $E(f)$ is countable. Here we use that the points of A_α are split in $\mathbb{A}_Y$.

We have $A_\alpha \cap E(f) = \big\{x \in A_\alpha : s(x) = \langle x, 0\rangle\big\} \cup \big\{x \in A_\alpha : s(x) = \langle x, 1\rangle\big\}$.

Look at the first set.

By continuity, if $s(x) = \langle x, 0\rangle$ then there is a rational interval (p_x, q_x) such that $x \in (p_x, q_x)$ and $s\big[(p_x, q_x)\big] \subseteq \big[\langle 0, 1\rangle, \langle x, 0\rangle\big]$.
And, clearly, if $x < y$ and $s(x) = \langle x, 0\rangle$ and $s(y) = \langle y, 0\rangle$ then $y \notin (p_x, q_x)$.

The first set is countable.
And, by symmetric reasoning, so is the second set.

So $A_\alpha \cap E(f)$ is countable, and therefore $E(f)$ is countable.

Verification

Step 2: $f\big[S(f)\big]$ is countable. Here we use that the points of V are split in $\mathbb{A}_Y$.

We look at $A_\alpha \cap C(f)$ and look at two subset:
$\big\{x : f(x) \in V \text{ and } s(x) = \langle f(x), 0\rangle\big\}$, and $\big\{x : f(x) \in V \text{ and } s(x) = \langle f(x), 1\rangle\big\}$.

As above we show that both sets are countable.

For x in the first set we take a rational interval (p_x, q_x) that contains x and that is mapped into $\big[\langle 0, 1\rangle, \langle f(x), 0\rangle\big]$.
In this case $f(x) < f(y)$ implies that $y \notin (p_x, q_x)$, and because f is injective on $C(f)$ the map $x \mapsto (p_x, q_x)$ is injective.

We find that $f\big[A_\alpha \cap C(f)\big] \cap V$ is countable.

By the conditions on the family $\{V\} \cup \{A_\beta : \beta \in \mathfrak{c}\}$ this implies that $f\big[S(f)\big]$ is countable.

Maybe we can use some stinking invariants after all ...

It turns out that we (Alan Dow and I) can find an explicit sequence $\langle K_n : n \in \omega \rangle$ of compact zero-dimensional spaces as in the beginning.

In 1983 Murray Bell constructed a sequence $\langle B_n : n \in \mathbb{N} \rangle$ of Boolean algebras such that, B_1 satisfies the countable chain condition and is not σ-2-linked, and for $n \geqslant 2$ the algebra B_n is σ-n-linked but not σ-$n+1$-linked.

Each B_n is in a natural way a subalgebra of $\mathcal{P}(\omega)/\mathit{fin}$.

Now let K_n be the one-point compactification of the topological sum $\bigoplus_{k \geqslant n} \mathrm{St}(B_k)$.

Maybe we can use some stinking invariants after all . . .

Clearly K_n maps onto K_{n+1}: map $\mathrm{St}(B_n) \cup \{\infty\}$ to the point ∞ and use the identity map everywhere else.

The space K_{n+1} does not map onto K_n, because this would produce a copy of the algebra B_n in the clopen algebra of K_{n+1}, which is σ-$n+1$-linked.

The space K_1 is a continuous image of ω^*, hence the sequence $\langle K_n : n \in \omega \rangle$ is as desired.

So there is a sequence of invariants that can help answer the original question after all.

Light reading

Murray Bell,
Two Boolean algebras with extreme cellular and compactness properties, Canadian Journal of Mathematics **35** (1983), no. 5, 824–838.

K. P. Hart,
Many subalgebras of $\mathcal{P}(\omega)/fin$, arXiv:2303.08491 [math.GN]