Closed copies of $\mathbb{N}$ in $\mathbb{R}^{\omega_1}$ and other realcompact spaces

Tá scéilín agam

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Credits

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Closed copies of $\mathbb{N}$ in realcompact spaces

Definitions

A subspace A of X is

- C-embedded if every continuous $f : A \to \mathbb{R}$ can be extended to a continuous $F : X \to \mathbb{R}$
- C^*-embedded if every *bounded* continuous $f : A \to \mathbb{R}$ can be extended to a continuous $F : X \to \mathbb{R}$

Closed copies of $\mathbb{N}$ in realcompact spaces

In *Some realcompact spaces* (Topology Proceedings **62** (2023), 205–216) we investigated the existence of closed copies of $\mathbb{N}$ in realcompact spaces that were C^*-embedded but not C-embedded.

Why?

Why not!

Or rather, because the known examples of (non-normal) spaces with closed subsets that were C^*-embedded but not C-embedded were pseudocompact.

Rather famous: Katětov's space P (called Λ by Gillman and Jerison in Exercise 6P).
It is $\beta\mathbb{R} \setminus (\beta\mathbb{N} \setminus \mathbb{N})$;
the space is pseudocompact, $\mathbb{N}$ is closed and C^*-embedded, but not C-embedded.

Closed copies of $\mathbb{N}$ in realcompact spaces

We asked for the minimum cardinals κ and κ^* such that

1. $\mathbb{R}^\kappa$ contains a closed copy of $\mathbb{N}$ that is C^*-embedded but not C-embedded
2. $\mathbb{R}^{\kappa^*}$ contains a closed copy of $\mathbb{N}$ that is not C^*-embedded

Using our examples we could show that $\kappa, \kappa^* \leqslant \mathfrak{c}$.

How?
Given X with a suitable copy of $\mathbb{N}$ consider the diagonal embedding of X into $\mathbb{R}^{C(X)}$.
Then (the image of) X is closed (defintion of realcompact) and C-embedded.
And (the image of) $\mathbb{N}$ retains the embedding properties it had but now with respect to $\mathbb{R}^{C(X)}$.
And for our X we have $\left|C(X)\right| = \mathfrak{c}$.

Closed copies of $\mathbb{N}$ in realcompact spaces

And YOU can show that $\kappa, \kappa^* > \aleph_0$ ...

And we already knew that $\kappa = \mathfrak{c}$ is consistent with $\neg$CH.

Closed copies of $\mathbb{N}$ in $\mathbb{R}^{\omega_1}$

After our paper was sent off into the world we found many examples of closed copies of $\mathbb{N}$ in $\mathbb{R}^{\omega_1}$ that were witnesses to $\kappa^* = \aleph_1$.

However . . .

BC: Before Covid

. . . Roman Pol informed us that

1. Keith M. Fox in 2013 already had many subsets of $\mathbb{N}^{\omega_1}$ that showed $\kappa^* = \aleph_1$
2. Elżbieta and Roman Pol in 2014 had another example for $\kappa^* = \aleph_1$ and an example showing $\kappa \leqslant \mathfrak{c}$
3. In 2020 Hirata and Yajima showed that $\mathfrak{r} > \aleph_1$ implies $\kappa > \aleph_1$ (and so $\kappa = \mathfrak{c}$ after adding $\aleph_2$ many random reals)

So . . . , the questions had been asked and answered before they were asked by us.

Nevertheless, it turned out that ignorance can be bliss.

Our examples are different from the others, and we found an extra consistency result.

Intermezzo: why $\aleph_1$?

Remember: in 1948 A. H. Stone showed that $\mathbb{N}^{\aleph_1}$ is not normal
(Exercise 2.3.E in Engelking's book; Problem 2.7.16 in the 1977 version).

So $\mathbb{N}^{\aleph_1}$ (and hence $\mathbb{R}^{\aleph_1}$) must have (many) instances where the Tietze-Urysohn extension theorem fails.

Can the domain of such an instance be as simple as possible: a closed copy of the discrete space $\mathbb{N}$?

In our ignorance we found a rich variety of examples, a general method, and the consistency of $\kappa = \aleph_1$ with all possible cardinal arithmetic.

Example 1: from an Aronszajn tree

Take an Aronszajn subtree, T, of ${}^{<\omega_1}\omega$, and an indexed family $\{A_t : t \in T\}$ of infinite subsets of $\mathbb{N}$ such that

1. $A_\emptyset = \mathbb{N}$,
2. $A_t \subseteq^* A_s$ whenever $s \subseteq t$,
3. $A_t = \bigcup_{n\in\omega} A_{t*n}$, and
4. $\{A_t : t \in T_\alpha\}$ is a partition of $\mathbb{N}$, whenever $\alpha \in \omega_1$.

Enumerate each level T_α as $\langle t(\alpha, n) : n \in \omega\rangle$, and write $A(\alpha, n) = A_{t(\alpha,n)}$.

Now define points $x_k \in \mathbb{R}^{\omega_1}$ ...

Example 1: from an Aronszajn tree

... as follows:

- $x_k(0) = 2^{-k}$, and
- $x_{2k}(\alpha) = x_{2k+1}(\alpha) = t(\alpha, m)$ iff $k \in A(\alpha, m)$

That's it.

This embeds $\mathbb{N}$ into $\mathbb{R} \times \mathbb{N}^{\aleph_1}$, in the guise of $\mathbb{R} \times \prod_{\alpha \geqslant 1} T_\alpha$.

Example 1: from an Aronszajn tree

The set $\{x_k : k \in \mathbb{N}\}$ is

- discrete because of coordinate 0,
- closed because an accumulation point would give you a branch in T, and
- not C^*-embedded because for every continuous function $f : \mathbb{R}^{\omega_1} \to [0,1]$ the closures of $f\big[\{x_{2k} : k \in \mathbb{N}\}\big]$ and $f\big[\{x_{2k+1} : k \in \mathbb{N}\}\big]$ intersect.

The last fact uses that such a continuous function factors through a countable subproduct, i.e., there is a α such that for all $x, y \in \mathbb{R}^{\omega_1}$ we have $x \restriction \alpha = y \restriction \alpha$ implies $f(x) = f(y)$. (Problem 2.7.12 in Engelking's book.)

Example 2: from an Aronszajn line

Let L be an Aronszajn line: an ordered continuum with the following properties

- first-countable,
- weight $\aleph_1$,
- no separable intervals, and
- every countable set has second-countable closure.

Let $\{x_\alpha : \alpha \in \omega_1\}$ be dense in L.

For each α let $K_\alpha = \mathrm{cl}\{x_\beta : \beta \leqslant \alpha\}$.
The K_αs form an increasing sequence of closed G_δ-sets and their union is equal to L.
Now ...

Example 2: from an Aronszajn line

... take a compactification $\gamma\mathbb{N}$ of $\mathbb{N}$ with remainder L.

Turns this into another compactification $\delta\mathbb{N}$ by identifying $\langle x, 0\rangle$ and $\langle x, 1\rangle$ in $\gamma\mathbb{N} \times \{0, 1\}$ whenever $x \in L$.

So: $\delta\mathbb{N}$ is a compactification of $\mathbb{N}$ with remainder L and such that L is the intersection of the closures of two complementary infinite subsets A and B of $\mathbb{N}$.

The K_α are zero-sets of $\delta\mathbb{N}$; use these to take continuous functions $f_\alpha : \delta\mathbb{N} \to [0, 1]$ such that $K_\alpha = f^{\leftarrow}[\{0\}]$ and $f_\alpha[\delta\mathbb{N} \setminus K_\alpha] \subseteq (0, 1)$ for all α.
The diagonal map pushes L into the boundary of $[0, 1]^{\omega_1}$, so the image of $\mathbb{N}$ will be closed in $(0, 1)^{\omega_1}$.
The sets A and B spoil the C^*-embedding, as above.
Discrete(?): at coordinate 0 map k to $\frac{1}{2} + 2^{-k-2}$ and L to $\{\frac{1}{2}\}$.

Example 3: from an Aronszajn tree

What we really used and needed was: a compact space with a cover by $\aleph_1$ many zero-sets that has no countable subcover.

The *path space* of an Aronszajn tree T, that is, the subspace of $\{0,1\}^T$ consisting of all (characteristic functions of) linearly ordered initial segments, is suitable input.
Zero-set number α: $\bigcap_{t \in T_\alpha} \{x : x(t) = 0\}$.
Hence: yet another example.

Example 4: from an injection of ω_1 into $\mathbb{R}$

Take an injective map $f : \omega_1 \to \mathbb{R}$ with the property that for every limit ordinal λ the set $\{f(\alpha) : \lambda \leqslant \alpha < \lambda + \omega\}$ is dense.

Let X be the subspace of $\{0,1\}^{\omega_1}$ that consists of the characteristic functions of the subsets on which f is monotonically increasing. (X is Corson compact.)

The space X has the required zero-set cover: for each limit λ let
$K_\lambda = \{x : x \restriction [\lambda, \lambda + \omega) \equiv 0\}$.
This example is different from the other three ...

Example 4: from an injection of ω_1 into $\mathbb{R}$

... not because we did not use an Aronszajn tree or line,
but because the cover $\{K_\lambda : \lambda \text{ a limit}\}$ is way more complicated than the other three (which were increasing).
You can show that if A and B are disjoint countable sets of limit ordinals then

$$\bigcap_{\lambda \in A} K_\lambda \setminus \bigcup_{\mu \in B} K_\mu$$

is nonempty

What can κ be?

There are examples of closed copies of $\mathbb{N}$ in $\mathbb{R}^{\mathfrak{c}}$ that are C^*-embedded, but not C-embedded.

After adding supercompact many Random reals one obtains a model in which *every C^*-embedded subset* of every space of character less than $\mathfrak{c}$ is C-embedded.

If $\mathfrak{r} > \aleph_1$ then *every C^*-embedded subset* of $\mathbb{R}^{\aleph_1}$ is C-embedded.
In particular $\kappa = \mathfrak{c} = \aleph_2$ in the $\aleph_2$-random real model.

So we cannot prove that κ is equal to $\aleph_1$.
Can we prove it consistent that $\kappa = \aleph_1$ (and CH fails of course)?

I thought you'd never ask ...

Models with $\kappa = \aleph_1$

Remember how we started Example 1: Take an Aronszajn subtree, T, of ${}^{<\omega_1}\omega$, and an indexed family $\{A_t : t \in T\}$ of infinite subsets of $\mathbb{N}$ such that

1. $A_\emptyset = \mathbb{N}$,
2. $A_t \subseteq^* A_s$ whenever $s \subseteq t$,
3. $A_t = \bigcup_{n\in\omega} A_{t*n}$, and
4. $\{A_t : t \in T_\alpha\}$ is a partition of $\mathbb{N}$, whenever $\alpha \in \omega_1$.

Now imagine this tree acts like an ultrafilter of character $\aleph_1$:

If $Y \subseteq \mathbb{N}$ then there is an ordinal α in ω_1 such that for every $t \in T_\alpha$ either $A_t \subseteq^* Y$ or $A_t \cap Y =^* \emptyset$.

Then ...

Models with $\kappa = \aleph_1$

... we can find a closed copy of $\mathbb{N}$ in $\mathbb{R}^{\omega_1}$ that is C^*-embedded but not C-embedded. (Take the embedding from example 1, but just do $x_k(\alpha) = t(\alpha, m)$ iff $k \in A(\alpha, m)$.)

If $Y \subseteq \mathbb{N}$ and α is as in our imagination then we can show that $\{x_k : k \in Y\}$ and $\{x_k : k \notin Y\}$ are already completely separated in the subproduct $\mathbb{R}^{\alpha+1}$.

And $x_k \mapsto k$ has no continuous extension to $\mathbb{R}^{\omega_1}$.

Models with $\kappa = \aleph_1$

Can we get this ultrafilter behaviour?

Glad you asked. Yes!

Everybody:
whip out your copy of Kunen's book (the original) and work Exercise VIII (A10), with some extra care.

What does that exercise ask you to do?
Construct an ultrafilter on $\mathbb{N}$ of character $\aleph_1$ by iterated forcing.
At stap α you obtain a new subset U_α of $\mathbb{N}$ that will be added to the base of the ultrafilter.
The extra care? Replicate U_α in al orderly fashion along the αth level of T.
(It needs a bit of bookkeeping.)

Models with $\kappa = \aleph_1$

Conclusion: $\kappa = \aleph_1$ is consistent with all (un)desirable cardinal arithmetic.

In particular $2^{\aleph_0}$ can be anything it ought to be.

Light reading

Alan Dow, K. P. Hart, Jan van Mill, Hans Vermeer
Closed copies of $\mathbb{N}$ *in* $\mathbb{R}^{\omega_1}$,
Topology and its Applications, in press (7 July 2025) 109541