An infinite library

Tá scéilín agam

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Regular but not completely regular

We make a regular space that is almost-but-not-quite completely regular.

Almost:

All closed subsets are G_δ-sets.

All points are zero-sets.

Not quite:

It is not completely regular

Why?

Guram Bezhanishvili had a remark and a question:
The lattice $O(X)$ of open subsets of a space X is the Dedekind-MacNeille completion of the lattice $Coz(X)$ of cozero-sets iff X is completely regular and its points are zero-sets.

Both conditions are used:
complete regularity gives: every open set is the supremum of a family of cozero-sets
points are zero-sets gives: every open set is the infimum of a family of cozero-sets
these two together work together to make $O(X)$ the D-M completion of $Coz(X)$

And the question is, of course:
Can we weaken complete regularity to mere regularity?
Or: if points are zero-sets in a regular space is the space then completely regular?

Many old examples do not work

Why?
Because they tend to have a closed set F and a point x outside F such that if f is continuous then there are quite a lot of points y in F such that $f(y) = f(x)$.

Exercise

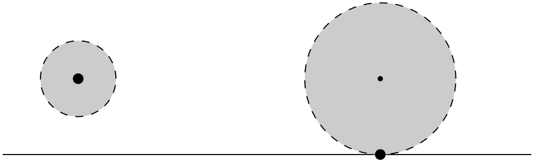
Verify this for Tychonoff's example of a regular space that is not completely regular (not the corkscrew in Steen and Seebach's book, but the accordeon in Tychonoff's original paper).

And many of the classical examples have many closed sets that are not G_δ.

The definition

On page 31 of Alexandroff and Hopf's *Topologie I*:

2°. R sei die Menge aller Punkte der Halbebene $y \geqq 0$ (es sind x und y Cartesische Koordinaten). Ist $p = (x, y)$ und $y > 0$, so soll jede offene Kreisscheibe mit dem Mittelpunkt p und einem Radius $< y$ Umgebung von p sein. Ist aber $y = 0$, so bestehe eine Umgebung von p aus dem Punkt p selbst und aus allen Punkten einer beliebigen offenen Kreisscheibe, die in der Halbebene $y > 0$ liegt und deren Randkreis die x-Achse im Punkte p berührt[1].



[1] Dieses Beispiel rührt von Herrn NIEMYTZKI her (vgl. § 6, Nr. 3).

Its properties

What happens in § 6, Nr. 3? In § 6 we find:

> **3. Beispiele.** Die in § 1, Nr. 1, unter 3° angeführte Konstruktion liefert, falls der Ausgangsraum *R* z. B. die Zahlengerade ist, einen topologischen Raum, welcher dem ersten, aber nicht dem zweiten Trennungsaxiom genügt. Der Raum von § 1, Nr. 4, 6°, ist ein Hausdorffscher irregulärer Raum, ebenso der Raum von § 1, Nr. 6, 1°. Der Raum von § 1, Nr. 6, 2°, ist regulär, jedoch nicht normal. In § 1, Nr. 4, 5°, ist ein Beispiel eines normalen nicht metrisierbaren Raumes gegeben.

So, the space R is regular but not normal.

Its properties

On the next page we get

Aufgabe. Man führe die Beweise der hier aufgestellten Behauptungen durch. Anweisung: die untrennbaren Mengen sind:

im Falle von § 1, Nr. 4, 6°, ein beliebiger Punkt (x_1, y_1) und die Menge aller Punkte (x_1, y), mit $y_1 < y < 1$;

im Falle § 1, Nr. 6, 1°, der Nullpunkt und die Menge D;

im Falle § 1, Nr. 6, 2°, die Menge der rationalen und die der irrationalen Punkte der x-Achse[1].

And in a footnote a further hint:

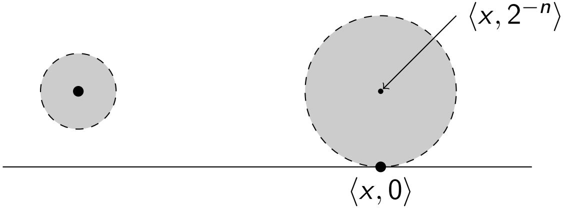
[1] Dieser letzte Fall ist etwas komplizierter als die übrigen; beim Beweis wird von der Tatsache Gebrauch gemacht, daß die Zahlengerade nicht als Summe von abzählbar-vielen nirgendsdichten Teilmengen dargestellt werden kann (Spezialfall des Baireschen Dichtigkeitssatzes, Kap. II, § 4, Satz V).

Basic neighbourhoods

We need the basic neighbourhoods of the points on the x-axis.
For $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we write

- $U'(x,n) = \{\langle u,v\rangle : \|\langle u,v\rangle - \langle x, 2^{-n}\rangle\| < 2^{-n}\}$ (the tangent disc of radius 2^{-n})
- $U(x,n) = \{\langle x,0\rangle\} \cup U'(x,n)$ (the nth neighbourhood of $\langle x,0\rangle$)

These are the sets that we will use frequently.



Non-normality

Two important sets (remember the Aufgabe):

- $P = \{\langle x, 0\rangle : x \text{ is irrational}\}$, and
- $Q = \{\langle x, 0\rangle : x \text{ is rational}\}$.

As we saw, these two closed sets cannot be separated by disjoint open sets.

Exercise

Prove this.

We shall study the behaviour of continuous fuctions on N,
and especially with respect to P and Q.

An elementary but useful Lemma

Lemma

Let $n \in \mathbb{N}$, let $I = [a, b]$ be an interval in $\mathbb{R}$, and let $D \subseteq I$ be dense in I. Then we have

$$\bigcup_{x \in D} U'(x, n) = \bigcup_{x \in I} U'(x, n)$$

Proof.

Proof by picture. □

Now prove: if O is open in N and $P \subseteq O$ then $\{q : \langle q, 0 \rangle \notin \operatorname{cl} O\}$ is nowhere dense in $\mathbb{Q}$ (normal topology).

First lemma

Lemma

Let $f : N \to [0,1]$ *be continuous and such that* $f(x,0) = 0$ *when* $x \in P$.
Then for every open interval I *in* $\mathbb{R}$ *we have*

$$\inf\{f(q,0) : q \in I \cap \mathbb{Q}\} = 0$$

Proof.

Let I and $\varepsilon > 0$ be given. By the Baire Category theorem there are a $k \in \mathbb{N}$ and an interval $J \subseteq I$ such that

$$D = \{x \in P \cap J : f[U(x,k)] \subseteq [0, \tfrac{1}{2}\varepsilon)\}$$

is dense in J (normal topology). Then $f(q,0) < \varepsilon$ for all $q \in J \cap \mathbb{Q}$. □

Second lemma

Lemma

Let $f : N \to [0,1]$ *be continuous and such that for every open interval* I *we have*

$$\inf\{f(q,0) : q \in I \cap \mathbb{Q}\} = 0$$

Then there is a dense G_δ*-set* G *in* $\mathbb{R}$ *(normal topology) such that* $f(x,0) = 0$ *whenever* $x \in G$*.*

Proof.

Let $k \in \mathbb{N}$. Let I be a (nonempty) open interval in $\mathbb{R}$.
Let $\varepsilon = 2^{-k}/3$ and cover $[0,1]$ by the open intervals $K_i = (i\varepsilon, (i+2)\varepsilon)$ where $i = -1, 0, \ldots, 3 \cdot 2^k - 1$. □

Second lemma

Proof, continued.

By the Baire Category theorem there are i and n in $\mathbb{N}$ and a (nonempty) open interval $J \subseteq I$ such that

$$D = \{x \in J : f[U(x,n)] \subseteq K_i\}$$

is dense in J (normal topology).
So, by the elementary but useful lemma, we get that $\bigcup_{x \in J} U'(x,n)$ is mapped into O_i.
Now take $q \in J \cap \mathbb{Q}$ with $f(q,0) < \varepsilon$.
But then $f(q,0) \in \operatorname{cl} O_i$ and so $i\varepsilon \leqslant f(q,0)$, we must have $i \leqslant 0$, and hence $\bigcup_{x \in J} U'(x,n)$ is actually mapped into $[0, 2\varepsilon)$.
It follows that $f(x,0) \leqslant 2\varepsilon < 2^{-k}$ for all $x \in J$.

We get a dense open set O_k in $\mathbb{R}$ (normal topology) such that $f(x,0) < 2^{-k}$ for all $x \in O_k$.
Now let k run free and take $G = \bigcap_{k \in \mathbb{N}} O_k$. □

If and only if

Note: the first lemma works not only for P but also for an arbitrary dense G_δ-set G.

This gives us a sort of "if and only if":
a continuous function $f : N \to [0,1]$ is zero on $\{\langle x, 0\rangle : x \in G\}$ for some dense G_δ-set G if and only if $\inf\{f(q,0) : q \in I \cap \mathbb{Q}\} = 0$ for every nonempty open interval I.

Reams of Niemytzki planes

Take infinitely many Niemytzki planes: $R = N \times \mathbb{N}$ (yes, R for ream),
with the product topology, where $\mathbb{N}$ is discrete.

Sew them together to form a book, as follows.

- For every even n and every irrational p identify the two points $\langle\langle p, 0\rangle, n\rangle$ and $\langle\langle p, 0\rangle, n+1\rangle$.
- For every odd n and every rational q identify the two points $\langle\langle q, 0\rangle, n\rangle$ and $\langle\langle q, 0\rangle, n+1\rangle$.

The resulting quotient space we call B (yes, B for book).
We let $\pi : R \to B$ be the quotient map.

Some easy properties

Exercise 1.5.H in Engelking's book

Every closed subset of N is a G_δ-set.

Lemma

The map π is perfect, hence B is regular.

Lemma

Every closed set in B is a G_δ-set.

Continuous functions on B

Let $f : B \to [0,1]$ be continuous and $F : R \to [0,1]$ the composition of π and f.

Lemma

If $F(x,0,0) = 0$ for all $x \in \mathbb{R}$ then there is a dense G_δ-set G in $\mathbb{R}$ (normal topology) such that for all $x \in G$ and all $n \in \mathbb{N}$ we have $F(x,0,n) = 0$.

Proof.

Apply the First and Second lemmas alternatingly to obtain, by recursion, for each even n a dense G_δ-set G_n such that $F(x,0,n+1) = F(x,0,n) = 0$ for all $x \in G_n$.
Then let $G = \bigcap_{n \text{ even}} G_n$. □

Add a point to R and B

Add a point ∞ to R, with basic neighbourhoods

$$U_m = \{\infty\} \cup \bigcup_{n \geqslant m} (N \times \{n\})$$

and add it to B also.
Give $B \cup \{\infty\}$ the quotient topology induced by the map π, extended to give $\pi(\infty) = \infty$.
The new map is still perfect, so $B \cup \{\infty\}$ is regular.

But $B \cup \{\infty\}$ is not completely regular.
For if $f : B \cup \{\infty\} \to [0,1]$ is continuous and equal to zero on the bottom leaf then by the last lemma we must have $f(\infty) = 0$ (in R the point ∞ is in the closure of $G \times \mathbb{N}$).

One book is not enough

The book-with-ornament $B \cup \{\infty\}$ comes close to giving us what we want.

In the space $B \cup \{\infty\}$ all closed sets are G_δ-sets.

The subspace B is in fact completely regular, and all points of B are zero-sets of $B \cup \{\infty\}$.

But $\{\infty\}$ is only a G_δ-set, not a zero-set.

Exercise

Prove this: if $f : B \to [0,1]$ is continuous then $\bigl\{x \in \mathbb{R} : (\forall n)(f(x,0,n) = f(\infty))\bigr\}$ contains a dense G_δ-set.

So ...

We build a library

. . . we take infinitely many books (eat your heart out Borges).

Start with $N \times \mathbb{N} \times \mathbb{N}$ and add a point ∞, with basic neighbourhoods

$$V_k = \bigcup_{m \geqslant k} \bigcup_{n \geqslant k} N \times \{\langle m, n \rangle\}$$

Then turn each column $N \times \{m\} \times \mathbb{N}$ into a book B_m, as above:

- identify $\big\langle \langle p, 0 \rangle, m, 2n \big\rangle$ and $\big\langle \langle p, 0 \rangle, m, 2n+1 \big\rangle$, when p is irrational, and
- identify $\big\langle \langle q, 0 \rangle, m, 2n+1 \big\rangle$ and $\big\langle \langle q, 0 \rangle, m, 2n+2 \big\rangle$, when q is rational

We build a library

The resulting space L (yes, L for library), is again regular and $L \setminus \{\infty\}$ is a collection of books, hence completely regular even.

As before all closed sets are G_δ in $L \setminus \{\infty\}$, and hence in L because $\{\infty\}$ is a G_δ too.

If we let $f : L \to [0, 1]$ be continuous and identically zero on all pages 0 of the books then we must have $f(\infty) = 0$ as well, because we get a single dense G_δ-set G in $\mathbb{R}$ (normal topology) such that $f(x, m, n) = 0$ for all $x \in G$ and all m and n.

But $\{\infty\}$ is in fact a zero-set because the function $f : L \to [0, 1]$, defined by

- $f(\infty) = 0$, and
- $f(x) = 2^{-m}$ if $x \in B_m$

is continuous and zero only at ∞.

A connected example

Actually, if we topologize $R \cup \{\infty\}$ slightly differently we can make do with just one book.

Basic neighbourhoods of $\{\infty\}$

$$W_k = \{\infty\} \cup \bigcup_{n \geqslant k} \big((k, \rightarrow) \times [0, \rightarrow) \times \{n\}\big)$$

If we now make a book we still cannot separate the zeroth leaf from ∞.

But $f : B \cup \{\infty\} \rightarrow [0, \infty)$ defined by $f(\infty) = 0$ and $f(x, y) = \mathrm{e}^{-x}$ shows that $\{\infty\}$ is a zero-set.

Light reading

K. P. Hart,
An Infinite Library, arXiv:2508.13325 [math.GN]