1 WE KNOW: A TRIVIAL CUT POINT IS NOT FAR
A NON-TRIVIAL CUT POINT IS FAR

a) A POINT THAT IS NOT FAR NEED NOT BE A TRIVIAL CUT POINT

LET $E_n = \{k\cdot 2^{-n} : 0 \le k \le 2^n\}$ ($n\in\omega$)
AND $E = \bigcup_{n\in\omega}(\{n\}\times E_n)$

LET $\mathcal{F} = \{E\setminus f : f \in \prod_{n\in\omega} E_n\}$.

THEN $\mathcal{F}$ HAS THE FINITE INTERSECTION PROPERTY AND EVERY $x\in\beta\mathbb{M}$ THAT EXTENDS $\mathcal{F}$ IS NOT A CUT POINT OF ITS $\mathbb{I}_u$.

b) A FAR POINT NEED NOT BE A CUT POINT.

LET $\mathcal{F} = \{F\subseteq \mathbb{M} : \lambda(\mathbb{M}\setminus F)<\infty;\ F \text{ IS CLOSED}\}$

- $\mathcal{F}$ HAS THE FINITE INTERSECTION PROPERTY
- IF $D\subseteq\mathbb{M}$ IS CLOSED AND DISCRETE THEN THERE IS AN $F\in\mathcal{F}$ THAT IS DISJOINT FROM D
- IF $x\in\beta\mathbb{M}$ EXTENDS $\mathcal{F}$ THEN x IS A FAR POINT THAT IS NOT A CUT POINT OF ITS $\mathbb{I}_u$.

THE POINT IS: IF $X\in x$ THEN $\lambda(X)=\infty$
SO IF $a\in A_x$ AND $b\in B_x$
THEN $\sum_{n\in\omega}(b_n-a_n)$ DIVERGES

ABOUT LAVER FORCING.

SOME TECHNICALITIES.

YOU CAN ORDER ${}^{<\omega}\omega$ LEXICOGRAPHICALLY
SAY $s<t$ IFF $s\subset t$ (PROPER INITIAL SEGMENT)
OR $s(n)<t(n)$ IF $n=\min\{k: s(k)\neq t(k)\}$.

MODIFY THIS ORDER: DIVIDE ${}^{<\omega}\omega$ INTO FINITE PIECES
$P_N = \{s : \max\{|s|, s(0),\dots,s(|s|-1)\} = N\}$ ($P_0=\{\emptyset\}$)

NEW ORDER $s \prec t$ IFF $s\in P_M$, $t\in P_N$ AND $M<N$
OR $s,t\in P_N$ AND $s<t$.

THE ORDER TYPE NOW IS ω
AND WE STILL HAVE
- IF $s\subset t$ THEN $s\prec t$
- IF $m<n$ THEN $s^\frown m \prec s^\frown n$.
- $\emptyset = \min {}^{<\omega}\omega$

IF $T \in \mathbb{L}$ SO, TO REPEAT

- $T \subseteq {}^{<\omega}\omega$ AND
- THERE IS AN $s_T \in T$ SUCH THAT
  $T = \{t : t \subseteq s_T \vee s_T \subseteq t\}$, AND
- IF $t \in T$ AND $s_T \subseteq t$ THEN
  $\mathrm{succ}_T(t) = \{k : t^\frown k \in T\}$
  IS INFINITE.

THEN $\{t \in T : s_T \subseteq t\}$ IS NATURALLY ISOMORPHIC TO ${}^{<\omega}\omega$ USING MONOTONE ENUMERATIONS EVERYWHERE:

- $\emptyset \longmapsto s_T$
- IF $s \longmapsto t$ THEN
  $s^\frown n \longmapsto t^\frown k_n$ ($\langle k_n : n \in \omega\rangle$ ENUMERATES $\{k : t^\frown k \in T\}$)

AND WE CONSIDER $\{t \in T : s_T \subseteq t\}$ ORDERED BY $\prec$ (OR $\prec_T$) VIA THIS ISOMORPHISM
AND $\langle T\langle n\rangle : n \in \omega\rangle$ IS THE CORRESPONDING ENUMERATION OF $\{t \in T : s_T \subseteq t\}$
[LAVER'S NOTATION]

$\longrightarrow$ WE ALWAYS USE THIS CANONICAL ENUMERATION.

EXTRA ORDERINGS: FOR $n \in \omega$ $\quad S \le^n T$ MEANS

$$S \le T \text{ AND } S\langle i\rangle = T\langle i\rangle \text{ FOR } i \le n$$

IN PARTICULAR $S \le^0 T$ MEANS $S \le T$ AND $s_S = s_T$

WHY? TO BUILD CONDITIONS/TREES WITH SPECIAL PROPERTIES.

LEMMA IF WE HAVE $\langle T_i : i \in \omega \setminus m\rangle$ IN $\mathbb{L}$ SUCH THAT $T_{i+1} \le^i T_i$ FOR $i \ge m$.
LET $T_\omega = \bigcup_{i \ge m} \{T_i\langle 0\rangle, \dots, T_i\langle i\rangle\} \cup \{s : s \subseteq T_m\langle 0\rangle\}$
THEN $T_\omega \in \mathbb{L}$ AND $T_\omega \le^i T_i$ FOR ALL i

- BECAUSE OF THE ISOMORPHISM EVERY SUCCESSOR SET IS VISITED INFINITELY OFTEN

NOTATION: IF $t \in T$ THEN
$T_t = \{s \in T : s \subseteq t \vee t \subseteq s\}$

SUPPOSE $T \in \mathbb{L}$, $X \in V$ IS FINITE
AND $T \Vdash \dot{x} \in X$

THEN THERE ARE AN $S \in \mathbb{L}$ AND $a \in X$
SUCH THAT $S \leq^{\circ} T$ AND $S \Vdash \dot{x} = a$

SUPPOSE NOT.

WE BUILD A BAD TREE R BELOW T

FIRST PUT $\{ t : t \leq s_T \}$ IN R

BY ASSUMPTION THERE IS NO $T' \leq^{\circ} T$
FOR WHICH THERE IS AN $a \in X$
SUCH THAT $T' \Vdash a = \dot{x}$

LOOK AT THE DIRECT SUCCESSORS OF s_T : $succ_T(s_T)$

LET $A = \{ t \in succ_T(s_T) : \exists T' \leq^{\circ} T_t \ \exists a \in X \ T' \Vdash \dot{x} = a \}$.

IF A IS INFINITE THEN WE HAVE
AN INFINITE SUBSET B AND ONE $a \in X$
SUCH THAT
FOR ALL $t \in A$ THERE IS $S_t \leq^{\circ} T_t$
SUCH THAT $S_t \Vdash \dot{x} = a$
BUT THEN $S = \bigcup_{t \in B} S_t \leq^{\circ} T$
AND $S \Vdash \dot{x} = a$

WE PUT B IN R

RECURSION IF t WAS PUT IN R THEN
THAT WAS BECAUSE THERE IS NO $T' \leq^{\circ} T_t$
FOR WHICH THERE IS AN $a \in X$
SUCH THAT $T' \Vdash \dot{x} = a$

SO ALL BUT FINITELY MANY DIRECT
SUCCESSORS OF t HAVE THIS PROPERTY
PUT THOSE IN R TOO ---

WE HAVE $R \leq^{\circ} T$

NOW TAKE $S \in R$ AND $a \in X$
SUCH THAT $S \Vdash \dot{x} = a$

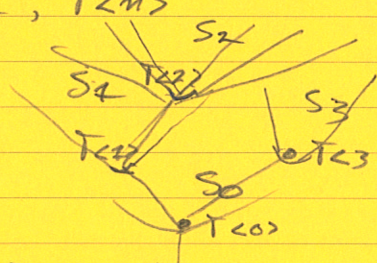
LET $t = s_S$

THEN $t \in R$ AND $S \leq^{\circ} R_t \leq^{\circ} T_t$

THERE IS OUR CONTRADICTION

IN GENERAL IF $T \in \mathbb{L}$, $X \in V$ IS FINITE
AND $T \Vdash \dot{x} \in X$
THEN FOR EVERY $n \in \omega$
THERE ARE $S \leq^n T$ AND $F \in [X]^{\leq n}$
SUCH THAT $S \Vdash \dot{x} \in F$.

LOOK AT THE POINTS $T\langle 0\rangle, \ldots, T\langle n\rangle$
THEY GIVE US A MAXIMAL
ANTICHAIN BELOW T:
TREES $S_0, S_1, \ldots, S_n$



WITH - $S_i\langle 0\rangle = T\langle i\rangle$ $(i \leq n)$

- $t \in S_i$ IFF $t \leq T\langle i\rangle$ OR $T\langle i\rangle \leq t$
<u>AND</u> IF $T\langle i\rangle < T\langle j\rangle$
THEN $T\langle j\rangle \not\leq t$

NOW APPLY THE PREVIOUS LEMMA TO EACH S_i
FIND $T_i \leq^0 S_i$ AND $a_i \in X$
SUCH THAT $T_i \Vdash \dot{x} = a_i$
THEN $S = \bigcup_{i \leq n} T_i$ AND $F = \{a_i : i \leq n\}$ WORK.

LIKEWISE - IF $T \Vdash \dot{x} \in V$
THEN THERE ARE $S \leq^0 T$ AND A
COUNTABLE SET $A \in V$ SUCH THAT $S \Vdash \dot{x} \in A$
- IF $T \Vdash$ "$\dot{A}$ IS COUNTABLE AND $\dot{A} \subseteq V$"
THEN FOR EACH n THERE IS A COUNTABLE
SET $B \in V$ WITH $S \leq^n T$ SUCH THAT
$S \Vdash \dot{A} \subseteq B$

LEMMA 6 OF LAVER FOR THE ITERATION
LET $\beta \leq \omega_1$ LET $p \in \mathbb{P}_\beta$ LET $F \subseteq \beta$
BE FINITE AND LET $n \in \omega$
LET $X \in V$ BE FINITE AND ASSUME $p \Vdash \dot{x} \in X$
THEN THERE ARE A $q \leq^n_F p$ AND $G \subseteq X$
SUCH THAT

- $q \Vdash \dot{x} \in G$
- $|G| \leq (n+1)^{|F|}$

WHERE $q \leq^n_F p$ MEANS WHAT IT SHOULD MEAN:
- $q \leq p$
- IF $\alpha \in F$ THEN $q \restriction \alpha \Vdash q(\alpha) \leq^n p(\alpha)$.

WHAT DOES LEMMA 6 DO FOR US!

THIS: IF $\langle X_n^\omega : n\in\omega\rangle$ IS A SEQUENCE OF FINITE SETS <u>IN V</u> AND $g\in\prod_{n\in\omega} X_n$ IS IN $V[G_{\omega_2}]$ THEN THERE IS A SEQUENCE $\langle Y_n : n\in\omega\rangle$ OF FINITE SETS WITH $Y_n\subseteq X_n$ AND $|Y_n|\le n$ FOR ALL n AND $g\in\prod_{n\in\omega} Y_n$.

HOW DO WE USE THIS!

WE HAVE A FAR POINT x IN V WE NEED AN f

① f IS THE GENERIC LAVER FUNCTION
NOTE: IF $T\in\mathbb{L}$ AND $t\in T$
THEN IF $t^\frown i\in T$ THEN $T_{t^\frown i}\Vdash \dot{f}(|t|)=i$

② ASSUME $g\in V[G_{\omega_2}]$ IS SUCH THAT $g(n)<f(n)$ FOR ALL n.
IN $V[f]$ LET $\dot{g}$ BE A NAME FOR g.
APPLY LEMMA 6 IN $V[f]$ TO GET A SEQUENCE $\langle F(n) : n\in\omega\rangle$ OF FINITE SUBSETS OF ω SUCH THAT
- $F(n)\subseteq f(n)$
- $|F(n)|\le n$
- $\Vdash(\forall n)(\dot{g}(n)\in F(n))$

③ IN V LET $\langle \dot{F}(n) : n\in\omega\rangle$ BE A NAME FOR $\langle F(n):n\in\omega\rangle$
SO $\Vdash_{\mathbb{L}}(\forall n)(\dot{F}(n)\subseteq\dot{f}(n)\wedge|\dot{F}(n)|=n)$

THE IDEA: THE UNIONS $\bigcup_{i\in F(n)}\left[\frac{i}{f(n)},\frac{i+1}{f(n)}\right]$
ARE VERY VERY THIN BECAUSE
$f(n)$ IS WAY BIGGER THAN n
IN FACT SO THIN THAT THEY LOOK
LIKE FINITE SETS OF POINTS
ALSO FROM THE PERSPECTIVE OF V
SO! x SHOULD BE ABLE TO AVOID IT.

PROBLEM: THESE UNIONS ARE NOT FIXED FROM V'S PERSPECTIVE; HOW DOES ON AVOID THESE THINGS WHEN THEY CAN BE ALL OVER THE PLACE?

APPLY OUR FIRST LEMMA, OFTEN.

LET $T \in \mathbb{L}$ AND $t \in T$

WE KNOW: IF $t^\frown i \in T$ THEN $T_{t^\frown i} \Vdash \dot{f}(|t|) = i$

NOW $[i]^{\le |t|}$ IS FINITE SO

WE CAN FIND $S \le^0 T_{t^\frown i}$ AND $H(t,i) \le i$ SUCH THAT

- $|H(t,i)| = |t|$
- $S \Vdash \dot{F}(|t|) = H(t,i)$.

(1) THIN OUT T SO THAT WE HAVE

$$H : \{\langle t,i\rangle : t^\frown i \in T\} \longrightarrow [\omega]^{<\omega}$$

WITH
- $|H(t,i)| = |t|$
- $H(t,i) \le i$
- $T_{t^\frown i} \Vdash \dot{F}(|t|) = H(t,i)$

(2) $H(t,i)$ DETERMINES A POINT IN $[0,1]^{|t|}$

$x(t,i) = \langle k/i : k \in H(t,i)\rangle$ (IN ORDER)

THE SET

$A_t = \{i : t^\frown i \in T\}$

HAS AN INFINITE SUBSET B_t SUCH THAT $\langle x(t,i) : i \in B\rangle$ CONVERGES TO A POINT $x(t)$ IN $[0,1]^{|t|}$

THIN OUT T SUCH THAT FOR EVERY t

$\langle x(t,i) : t^\frown i \in T\rangle$ CONVERGES TO $x(t)$.