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C-EMBEDDING, LINDELÖFNESS, AND ČECH-COMPLETENESS

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In memory of Gary Gruenhage

Abstract. We show that in the class of Lindelöf Čech-complete spaces the property of being C-embedded is quite well-behaved. It admits a useful characterization that can be used to show that products and perfect preimages of C-embedded spaces are again C-embedded. We also show that both properties, Lindelöf and Čech-complete, are needed in the product result.

Introduction

In [2] we investigated whether in realcompact spaces there could be closed, countable, and discrete subspaces (closed copies of the space $\mathbb{N}$ of natural numbers) that were C^*-embedded but not C-embedded, or even not C^*-embedded. In the follow-up paper [3] we looked for the smallest power of the real line $\mathbb{R}$ that could contain such closed copies of $\mathbb{N}$.

In the present paper we consider more general spaces. It appears that the members of the class of Lindelöf Čech-complete spaces behave much like $\mathbb{N}$ as regards C-embedding. Our positive results characterize C-embedding and allow us to conclude that, in this class, C-embedding is preserved by products and perfect preimages. We also show, by means of

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examples, that neither assumption, Lindelöfness nor Čech-completeness, can be dropped in these results.

1. Preliminaries

All spaces in this paper are assumed to be, at least, Tychonoff spaces. The books [4, 5] are our primary sources for all undefined topological notions.

1.1. **C-embedding.** It behoves us to define the central notion of this paper, that of C-embedding.

A subspace A of a space X is said to be C-embedded in X if every continuous function from A to $\mathbb{R}$ admits a continuous extension to X.

In [5, Theorem 1.18], it is shown that A is C-embedded in X if and only if

(1) it is C^*-embedded in X, that is, every *bounded* continuous function from A to $\mathbb{R}$ admits a continuous extension to X, and
(2) every zero-set Z that is disjoint from A is *completely separated* from A, that is, there is a continuous function $f : X \to \mathbb{R}$ such that $f(a) = 0$ when $a \in A$ and $f(z) = 1$ when $z \in Z$.

We shall use this equivalence in our proofs as well as the following characterization of (1): if Z_1 and Z_2 are disjoint zero-sets of A then they are completely separated in X, see [5, Theorem 1.17].

Also, given this equivalence one can weaken (1) to: A is z-embedded, meaning that for every zero-set Z of A, there is a zero-set Z^+ of X such that $Z = A \cap Z^+$. The point is that if Z_1 and Z_2 are disjoint zero-sets of A then the intersection $Z_1^+ \cap Z_2^+$ is a zero-set that is disjoint from A; then (2) lets us make Z_1^+ and Z_2^+ a bit smaller so that they become disjoint.

Below we freely use the diagonal embedding e of a space X into $\mathbb{R}^{C(X)}$, defined by $e(x) = \langle f(x) : f \in C(X) \rangle$. It is well known that $e[X]$ is C-embedded in the product and that $e[X]$ is closed whenever X is realcompact. Our examples are all realcompact, either because they are Lindelöf, or discrete and of small enough cardinality. We recommend [5, Chapter 8] and [4, Section 3.11] for basic information on realcompactness.

1.2. **Rationals and irrationals.** A few of the examples in Section 3 use some facts about the spaces of rational and irrational numbers, and completely metrizable spaces, that we record here.

We let $\mathcal{N}$ denote the zero-dimensional Baire space $\mathbb{N}^{\mathbb{N}}$, the product of countably many copies of the discrete space $\mathbb{N}$, denoted $B(\aleph_0)$ in [4, Example 4.2.12]. In [4, Exercises 4.3.G and 4.3.H] we find the results that we shall use below: $\mathcal{N}$ is homeomorphic to the subspace of irrational numbers in $\mathbb{R}$, and any two countable dense subsets of $\mathbb{R}$ can be mapped to each other by an autohomeomorphism of $\mathbb{R}$.

1.3. **Two technical results.** In our examples we use Lavrentieff's theorem, [4, Theorem 4.3.21], which states that if X and Y are completely metrizable, with subspaces A and B respectively, and $f : A \to B$ is a homeomorphism then f has an extension to a homeomorphism $\tilde{f} : \tilde{A} \to \tilde{B}$, where $\tilde{A}$ and $\tilde{B}$ are G_δ-sets.

We also use a result due to various authors, [4, Problem 2.7.12 (d)]: Let κ be an infinite cardinal and let $f : X \to \mathbb{R}$ be continuous, where X is a product of a sequence $\langle X_\alpha : \alpha < \kappa \rangle$ of separable spaces. Then there are a countable subset E of κ and a continuous function $g : \prod_{\alpha\in E} X_\alpha \to \mathbb{R}$ such that $f = g \circ \pi_E$, where $\pi_E : X \to \prod_{\alpha\in E} X_\alpha$ is the projection — in words: f factors through a countable subproduct.

2. Positive results

We begin by giving an external characterization of closed subspaces of Tychonoff spaces that are both Lindelöf and C-embedded.

The following lemma characterizes C-embeddedness for arbitrary closed subsets.

Lemma 2.1. *Let A be a closed subset of a space X. Then the following three conditions are equivalent.*

(1) *A is C-embedded in X.*
(2) *A is z-embedded in X and for every zero-set Z of $\mathrm{cl}_{\beta X} A$ that is disjoint from A there is a zero-set Z^+ of βX that is disjoint from X and such that $Z = Z^+ \cap \mathrm{cl}_{\beta X} A$.*
(3) *A is z-embedded in X and for every zero-set Z of $\mathrm{cl}_{\beta X} A$ that is disjoint from A there is a countable family $\mathcal{Z}$ of zero-sets of βX such that $Z \subseteq \bigcup \mathcal{Z} \subseteq \beta X \setminus X$.*

Proof. To prove that (1) implies (2) we take a zero-set Z of $\mathrm{cl}_{\beta X} A$ and construct Z^+, as follows. Let $f : \mathrm{cl}_{\beta X} A \to [0,1]$ be continuous such that $Z = \{a \in \mathrm{cl}_{\beta X} A : f(a) = 0\}$ and consider its restriction $f \restriction A$ to A; this is a function from A to $(0,1]$. By C-embeddedness we have an extension $F : X \to (0,1]$ of $f \restriction A$, which we then extend to $\beta F : \beta X \to [0,1]$. Then let Z^+ be the zero-set of βF.

That (2) implies (3) is clear so we turn to proving that (3) implies (1).

We already know that A is z-embedded in X, so let Z be a zero-set of X that is disjoint from A; we show that Z and A are completely separated. Let Z_β be a zero-set of βX such that $Z = X \cap Z_\beta$ and let $Z_A = Z_\beta \cap \mathrm{cl}_{\beta X} A$. Then Z_A is a zero-set of $\mathrm{cl}_{\beta X} A$ that is disjoint from A.

Let $\mathcal{Z}$ be a countable family of zero-sets of βX as in the assumption. Say $\mathcal{Z} = \{Z_n : n \in \omega\}$, and for every n let $C_n = \beta X \setminus Z_n$ be the complementary cozero-set.

Then $X \subseteq L = \bigcap_{n\in\omega} C_n$, and by [7, Lemma 2.2] (or [4, Exercise 3.8.F]) the space L is Lindelöf. In addition the sets $Z_\beta \cap L$ and $\mathrm{cl}_{\beta X} A \cap L$ are closed and disjoint in L; as L is normal these sets are completely separated in L, and so Z and A are completely separated in X, because $X \subseteq L \subseteq \beta X$. □

Using this lemma we get our principal result.

Theorem 2.2. *Let A be a closed subset of a space X. Then the following three conditions are equivalent.*

(1) *A is Lindelöf and C-embedded in X.*
(2) *For every compact subset K of $\mathrm{cl}_{\beta X} A \setminus A$ there is a zero-set Z of βX such that $K \subseteq Z \subseteq \beta X \setminus X$.*
(3) *For every compact subset K of $\mathrm{cl}_{\beta X} A \setminus A$ there is a countable family $\mathcal{Z}$ of zero-sets of βX such that $K \subseteq \bigcup \mathcal{Z} \subseteq \beta X \setminus X$.*

Proof. To prove that (1) implies (2) we let K be a compact subset of $\mathrm{cl}_{\beta X} A \setminus A$ and find a zero-set Z of βX such that $K \subseteq Z \subseteq \beta X \setminus X$.

To begin we choose for every $a \in A$ a continuous function $f_a : \beta X \to [0,1]$ such that $f_a(a) = 1$ and $f_a(x) = 0$ if $x \in K$. For each a we let $U_a = f^{\leftarrow}\bigl[(\frac{1}{2}, 1]\bigr]$. There is a countable subset $\{a_n : n \in \omega\}$ of A such that $A \subseteq \bigcup_{n\in\omega} U_{a_n}$.

Let $g = \sum_{n\in\omega} 2^{-n} f_{a_n}$. Then g is continuous, $g(a) > 0$ when $a \in A$, and $g(x) = 0$ when $x \in K$. We would like to let $Z = g^{\leftarrow}(0)$, but that set may intersect X. However, $S = \{x \in X : g(x) = 0\}$ is a zero-set of X that is disjoint from A. As A is C-embedded in X the sets S and A are completely separated. Let $f : X \to [0,1]$ be continuous such that $f(x) = 1$ if $x \in S$ and $f(a) = 0$ if $a \in A$. Note that βf vanishes on $\mathrm{cl}_{\beta X} A$ and in particular on K.

Now let $h = g + \beta f$. Then $h(x) \geqslant g(x) > 0$ when $x \in X \setminus S$ and $h(x) \geqslant 1$ when $x \in S$. Also, $h(x) = 0$ when $x \in K$. It follows that $h^{\leftarrow}(0)$ is the zero-set of βX that we seek.

Clearly (2) implies (3).

We finish by proving that (3) implies (1). To begin: the present condition (3) is stronger than the second part of (3) in Lemma 2.1. We need to show that A is Lindelöf and z-embedded in X. It will actually be simpler to show that A is C^*-embedded in $\mathrm{cl}_{\beta X} A$.

That A is Lindelöf is proved as follows. Let $\mathcal{U}$ be a collection of open subsets of βX that covers A. Let $K = \mathrm{cl}_{\beta X} A \setminus \bigcup\mathcal{U}$ and let $\mathcal{Z}$ be a countable family of zero-sets of βX as in the assumption.

As in the proof of Lemma 2.1 the complement L of $\bigcup \mathcal{Z}$ in βX is Lindelöf and it contains X. Then $L \cap \operatorname{cl}_{\beta X} A$ is Lindelöf as well and, moreover, contained in $\bigcup \mathcal{U}$. But $A \subseteq L \cap \operatorname{cl}_{\beta X} A$, so A is covered by a countable subfamily of $\mathcal{U}$.

To see that A is C^*-embedded in $\operatorname{cl}_{\beta X} A$, and hence in βX, we show that if E and F are disjoint closed subsets of A then their closures in $\operatorname{cl}_{\beta X} A$ are disjoint; this shows that $\operatorname{cl}_{\beta X} A$ actually *is* the Čech-Stone compactification of the normal space A.

Let E and F be as above and let $K = \operatorname{cl}_{\beta X} E \cap \operatorname{cl}_{\beta X} F$. Then $K \subseteq \operatorname{cl}_{\beta X} A \setminus A$ and hence there is a countable family $\mathcal{Z}$ of zero-sets of βX as in our assumption.

We have just established that $L = \beta X \setminus \bigcup \mathcal{Z}$ is Lindelöf, hence L is normal as well. Also $X \subseteq L \subseteq \beta X$, and so $\beta L = \beta X$.

In addition we have $\operatorname{cl}_L E \cap \operatorname{cl}_L F = \emptyset$ and hence $\operatorname{cl}_{\beta X} E \cap \operatorname{cl}_{\beta X} F = \emptyset$ (so in hindsight $K = \emptyset$). □

There are two special cases of this result that are worth recording here. They consider Lindelöf subspaces that are locally compact or Čech-complete.

Theorem 2.3. *Let A be a closed and locally compact subset of X. Then the following three conditions are equivalent.*

(1) *A is Lindelöf and C-embedded in X.*
(2) *There is a zero-set Z of βX such that* $\operatorname{cl}_{\beta X} A \setminus A \subseteq Z \subseteq \beta X \setminus X$.
(3) *There is a countable family $\mathcal{Z}$ of zero-sets of βX such that* $\operatorname{cl}_{\beta X} A \setminus A \subseteq \bigcup \mathcal{Z} \subseteq \beta X \setminus X$.

Proof. The set $\operatorname{cl}_{\beta X} A \setminus A$ is closed and hence compact. Therefore it is necessary and sufficient to assume or establish (2) and (3) for that set only. □

Theorem 2.4. *Let A be a closed and Čech-complete subset of X. Then the following conditions are equivalent.*

(1) *A is Lindelöf and C-embedded in X.*
(2) *There is a countable family $\mathcal{Z}$ of zero-sets of βX such that* $\operatorname{cl}_{\beta X} A \setminus A \subseteq \bigcup \mathcal{Z} \subseteq \beta X \setminus X$.

Proof. By the definition of Čech-completeness the set $\operatorname{cl}_{\beta X} A \setminus A$ is an F_σ-subset of $\operatorname{cl}_{\beta X} A$. One applies (2) or (3) in Theorem 2.2 to the countably many closed sets whose union is $\operatorname{cl}_{\beta X} A \setminus A$ to obtain the desired cover. □

From these characterizations we deduce two results about the preservation of C-embeddedness.

Theorem 2.5. *Let $\langle X_i : i < k\rangle$ be a sequence of spaces, where k is a finite ordinal or ω, and let $\langle A_i : i < k\rangle$ be a corresponding sequence of C-embedded subspaces (A_i of X_i) that are closed, Lindelöf, and Čech-complete. Then the product $\prod_{i<k} A_i$ is Lindelöf, Čech-complete, and C-embedded in $\prod_{i<k} X_i$.*

Proof. We write $A = \prod_{i<k} A_i$ and $X = \prod_{i<k} X_i$.

For each i let $\mathcal{Z}_i$ be a countable family of zero-sets in βX_i such that

$$\operatorname{cl}_{\beta X_i} A_i \setminus A_i \subseteq \bigcup \mathcal{Z}_i \subseteq \beta X_i \setminus X_i.$$

Then $\operatorname{cl} A \setminus A$ is covered by union $\mathcal{Z}$ of the families $\{\pi_i^{\leftarrow}[Z] : Z \in \mathcal{Z}_i\}$. These are countable families of zero-sets in $\prod_{i<k} \beta X_i$ and their members are contained in $(\prod_{i<k} \beta X_i) \setminus X$.

Let $f : \beta X \to \prod_{i<k} \beta X_i$ be the natural map. Then $\{f^{\leftarrow}[Z] : Z \in \mathcal{Z}\}$ is a countable family of zero-sets in βX. Because f is perfect its union $\bigcup \mathcal{Z}$ is contained in $\beta X \setminus X$ and it contains $\operatorname{cl}_{\beta X} A \setminus A$.

Theorem 2.4 implies that A is Lindelöf and C-embedded in X. □

By strengthening the assumptions on the subspaces and weakening the conclusion we get a version of this result for arbitrary products.

Corollary 2.6. *Let $\langle X_\alpha : \alpha < \kappa\rangle$ be an arbitray sequence of spaces, and let $\langle A_\alpha : \alpha < \kappa\rangle$ be a corresponding sequence of C-embedded subspaces (A_α of X_α) that are closed, Lindelöf, Čech-complete, and separable. Then the product $A = \prod_{\alpha<\kappa} A_\alpha$ is C-embedded in $X = \prod_{\alpha<\kappa} X_\alpha$.*

Proof. If $f : A \to \mathbb{R}$ is continuous then, by separability of the factors, the factorization result from section 1.3 implies that f factors through a countable subproduct $\prod_{\alpha\in E} A_\alpha$. The previous theorem implies that the factored map f_E has a continuous extension F to $\prod_{\alpha\in E} X_\alpha$. Then F determines a continous extension of f to X. □

Theorem 2.7. *Let A be a closed, Lindelöf and Čech-complete subspace of X that is also C-embedded and let $f : Y \to X$ be a perfect surjection. Then $f^{\leftarrow}[A]$ is C-embedded in Y.*

Proof. The previous proof applies. If $\mathcal{Z}$ is a countable family of zero-sets of βX such that

$$\operatorname{cl}_{\beta X} A \setminus A \subseteq \bigcup \mathcal{Z} \subseteq \beta X \setminus X$$

then, because f is perfect, we have

$$\operatorname{cl}_{\beta Y} f^{\leftarrow}[A] \setminus f^{\leftarrow}[A] \subseteq \bigcup\{f^{\leftarrow}[Z] : Z \in \mathcal{Z}\} \subseteq \beta Y \setminus Y. \quad \square$$

Remark 2.8. The proofs that (1) implies (2) and (3) implies (1) in Theorem 2.2 use properties of βX.

The proof of Theorem 2.5 shows, implicitly, that if A satisfies condition (3) in *some* compactification of X then it will satisfy that condition in βX as well. The converse of this does not hold.

For let X be an uncountable discrete space, and A a countable subset of X. Then A is — trivially — Lindelöf, Čech-complete, and C-embedded in X, and, in addition, $\mathrm{cl}_{\beta X} A \setminus A$ is itself a zero-set of βX.

But A does not satisfy condition (3) in the one-point compactification αX of X. Indeed $\mathrm{cl}_{\alpha X} A \setminus A = \alpha X \setminus X = \{\infty\}$, where ∞ is the point at infinity, and this remainder contains no zero-set of αX.

3. Examples

An easy consequence of Theorem 2.5 is that if two spaces X and Y contain closed copies, N_1 and N_2 respectively, of $\mathbb{N}$ that are C-embedded then the product $N_1 \times N_2$ is C-embedded in $X \times Y$.

This can also be established in an elementary way. There are continuous functions $f_1 : X \to \mathbb{R}$ and $f_2 : Y \to \mathbb{R}$ such that f_1 maps N_1 injectively into $\{2^n : n \in \mathbb{N}\}$ and f_2 maps N_2 injectively into $\{3^n : n \in \mathbb{N}\}$. Then $f : X \times Y \to \mathbb{R}$, defined by $f(x, y) = f_1(x) \cdot f_2(y)$, maps $N_1 \times N_2$ injectively into $\mathbb{N}$ and this suffices to ensure C-embedding.

The countability of $\mathbb{N}$ corresponds to the Lindelöf assumption in Theorem 2.5. This assumption cannot be dropped completely as the next example shows.

Example 3.1. Let κ be a cardinal such that there is an uncountable closed and discrete subset D that is C-embedded in $\mathbb{R}^\kappa$. Then $D \times D$ is not C^*-embedded in $\mathbb{R}^\kappa \times \mathbb{R}^\kappa$.

Remark 3.2. We can have D of cardinality $\aleph_1$ and with $\kappa = 2^{\aleph_1}$. For if λ is less than the first measurable cardinal then λ, with its discrete topology, is realcompact and so the image of λ under the diagonal map $e : \lambda \to \mathbb{R}^{C(\lambda)}$ is closed and C-embedded.

Proof of Example 3.1. We define $f : D \times D \to [0, 1]$ as follows:

- $f(d, e) = 0$ if $d \neq e$, and
- $d \mapsto f(d, d)$ maps into the interval $(0, 1]$, injectively if $|D| \leqslant \mathfrak{c}$, and surjectively if $|D| \geqslant \mathfrak{c}$ (and so bijectively if $|D| = \mathfrak{c}$).

Because $D \times D$ is discrete f is automatically continuous.

Now assume $F : \mathbb{R}^\kappa \times \mathbb{R}^\kappa \to \mathbb{R}$ is a continuous extension of f and let C be a countable subset of κ such that F factors through $\mathbb{R}^C \times \mathbb{R}^C$. So we have a continuous map $g : \mathbb{R}^C \times \mathbb{R}^C \to [0, 1]$ such that $F = g \circ (\pi \times \pi)$ where $\pi : \mathbb{R}^\kappa \to \mathbb{R}^C$ is the projection.

Let $E = \pi[D]$ and observe that E is uncountable. If $|D| \leqslant \mathfrak{c}$ then π is a bijection between D and E because $d \mapsto g(\pi(d), \pi(d))$ is injective, and if $|D| \geqslant \mathfrak{c}$ then $g[E \times E] = [0,1]$, so $|E| \geqslant \mathfrak{c}$ as well.

Now let $e \in E$, then $g(e,e) > 0$ and so there is a neighbourhood U of e in $\mathbb{R}^C$ such that $g(x,y) > 0$ for all $x, y \in U$. We claim that $U \cap E = \{e\}$. Indeed, if $x \in U$ and $x \neq e$ then $g(x,e) > 0$ whereas $g(y,e) = 0$ whenever $y \in E$ and $y \neq e$.

It follows that E is an uncountable relatively discrete subset of the separable and metrizable space $\mathbb{R}^C \times \mathbb{R}^C$, a clear impossibility. $\square$

The assumption that the factors be Čech-complete cannot be dropped either.

Theorem 3.3. *The product $\mathbb{Q} \times \mathbb{Q}$ is not C^*-embedded in $\mathbb{R}^{C(\mathbb{Q})} \times \mathbb{R}^{C(\mathbb{Q})}$, where $\mathbb{Q}$ is embedded in $\mathbb{R}^{C(\mathbb{Q})}$ via the diagonal embedding $e : \mathbb{Q} \to \mathbb{R}^{C(\mathbb{Q})}$.*

Before we give the proof we need a lemma first.

Lemma 3.4. *Let X be a separable and metrizable space, and let $f : X \times X \to \mathbb{R}$ be a continuous function. Then the following two statements about f are equivalent.*

1. *f has a continuous extension to the product $\mathbb{R}^{C(X)} \times \mathbb{R}^{C(X)}$, where we identify X with its image $e[X]$ under the diagonal embedding $e : X \to \mathbb{R}^{C(X)}$, and*
2. *there is a completely metrizable extension M of X such that f has a continuous extension to $M \times M$.*

Proof. Necessity: assume $F : \mathbb{R}^{C(X)} \times \mathbb{R}^{C(X)} \to \mathbb{R}$ is an extension of f. We can find a countable subset E of $C(X)$ such that F factors through the partial product $\mathbb{R}^E \times \mathbb{R}^E$. We can enlarge E, if need be, so that the projection $\pi_E : \mathbb{R}^{C(X)} \to \mathbb{R}^E$ is a homeomorphism on $e[X]$, that is, $\pi_E \circ e : X \to \mathbb{R}^E$ is a homeomorphic embedding; here is where we use that X is regular and second-countable.

Now let $G : \mathbb{R}^E \times \mathbb{R}^E \to \mathbb{R}$ be such that $F = G \circ (\pi_E \times \pi_E)$. Then $\mathbb{R}^E$ is a completely metrizable extension of X, and G is a continuous extension of f.

Sufficiency: assume M is a completely metrizable space that contains X and such that there is a continuous extension $g : M \times M \to \mathbb{R}$ of f. We assume X is dense in M as its closure in M is completely metrizable as well.

Then there is an embedding $h : M \to \mathbb{R}^\omega$ such that $h[M]$ is closed. Because $h[M] \times h[M]$ is closed in $\mathbb{R}^\omega \times \mathbb{R}^\omega$ there is a continuous function $G : \mathbb{R}^\omega \times \mathbb{R}^\omega \to \mathbb{R}$ that extends g (more precisely: such that $g = G \circ h$). The countably many projections $\pi_n : \mathbb{R}^\omega \to \mathbb{R}$ yield members of $C(M)$ via $p_n = \pi_n \circ h$. These give us a projection $\Pi : \mathbb{R}^{C(X)} \to \mathbb{R}^\omega$ such that $\Pi \circ e = h$ on X. Then $G \circ (\Pi \times \Pi)$ is the extension of f to $\mathbb{R}^{C(X)} \times \mathbb{R}^{C(X)}$. □

Proof of Theorem 3.3. By the lemma, to show that $\mathbb{Q} \times \mathbb{Q}$ is not C^*-embedded in $\mathbb{R}^{C(\mathbb{Q})} \times \mathbb{R}^{C(\mathbb{Q})}$ it suffices to exhibit a bounded continuous function $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ that has no continuous extension to $M \times M$ whenever M is a completely metrizable extension of $\mathbb{Q}$.

We claim that it suffices to find a continuous function $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ such that there is no G_δ-subset G of $\mathbb{R}$ such that f has a continuous extension to $G \times G$. Indeed, if M is an arbitrary completely metrizable extension of $\mathbb{Q}$, say with embedding $g : \mathbb{Q} \to M$, then Lavrentieff's theorem yields G_δ-sets A in $\mathbb{R}$ and B in X, and a homeomorphism $\bar{g} : A \to B$ that extends g. If $\bar{f}$ were a continuous extension of f to $M \times M$ then $\bar{f} \circ (\bar{g} \times \bar{g})$ would be a continuous extension of f to $A \times A$.

To define f we let L be the line in the plane with equation $y = x + \pi$. Clearly L is disjoint from $\mathbb{Q} \times \mathbb{Q}$. But, if A is a G_δ-subset of R that contains $\mathbb{Q}$ then $(A \times A) \cap L \neq \emptyset$. For let A be such a G_δ-set then both A and $A - \pi$ are dense G_δ-subsets of $\mathbb{R}$ and hence, by the Baire Category theorem the intersection $B = A \cap (A - \pi)$ is also a dense G_δ-set. But if $x \in B$ then $(x, x + \pi) \in (A \times A) \cap L$.

Now define $f : \mathbb{Q} \times \mathbb{Q} \to [-1, 1]$ by

$$f(p, q) = \frac{q - p - \pi}{|q - p - \pi|}$$

Then f has no continuous extension to any point of L. □

Example 3.5. One may wonder whether Theorem 3.3 can be proved using a homeomorphism between $\mathbb{Q} \times \mathbb{Q}$ and $\mathbb{Q}$. The idea is that such a homeomorphism should change the geometry of $\mathbb{Q} \times \mathbb{Q}$ too much to allow it to be extended to $\mathbb{R}^{C(\mathbb{Q})} \times \mathbb{R}^{C(\mathbb{Q})}$.

We exhibit two homeomorphisms between $\mathbb{Q} \times \mathbb{Q}$ and $\mathbb{Q}$: one that can be extended and one that cannot.

The first comes via a direct application of Lemma 3.4.

We let $\mathcal{N} = \mathbb{N}^\mathbb{N}$ be zero-dimensional Baire space and let $\mathbb{Q}$ be embedded in $\mathcal{N}$ as the subset Q of sequences that end in zeros, so $Q = \{x : (\exists m)(\forall n \geqslant m)(x_n = 0)\}$.

Now, the homeomorphism $h : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ is obtained by interleaving sequences and it maps $Q \times Q$ to Q. If we compose this map with an embedding g of $\mathcal{N}$ into $\mathbb{R}$ that maps Q onto $\mathbb{Q}$ then $g \circ h$ is an extension to $\mathcal{N} \times \mathcal{N}$ of its restriction to $Q \times Q$.

To obtain the second homeomorphism we take the embedding $g : \mathcal{N} \to \mathbb{R}$ used above with $e[Q] = \mathbb{Q}$ and we let $N = g[\mathcal{N}]$.

Then N is a G_δ-set that contains $\mathbb{Q}$ and the composition $G = g \circ h \circ (g^{-1} \times g^{-1})$ is a homeomorphism from $N \times N$ to N. In particular, G is injective on the intersection of $N \times N$ with the line L above.

Next define a homeomorphism $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q} \times \mathbb{Q}$ by $f(p, q) = (p + 1, q + 1)$ if $q > p + \pi$ and $f(p, q) = (p - 1, q - 1)$ if $q < p + \pi$. Then the composition $G \circ f$ is still a homeomorphism between $\mathbb{Q} \times \mathbb{Q}$ and $\mathbb{Q}$. However, if M is a G_δ-set that contains $\mathbb{Q}$ then $L \cap ((M \cap N) \times (M \cap N))$ is nonempty and $G \circ f$ has no continuous extension to any point in that intersection.

For let $\langle x, y\rangle$ be a point in the intersection and take two sequences $\bigl\langle\langle p_n, q_n\rangle : n \in \omega\bigr\rangle$ and $\bigl\langle\langle r_n, s_n\rangle : n \in \omega\bigr\rangle$ in $\mathbb{Q} \times \mathbb{Q}$ that converge to $\langle x, y\rangle$ and such that $q_n > p_n + \pi$ and $s_n < r_n + \pi$ for all n. Then $\lim_n(G \circ f)(p_n, q_n) = G(x + 1, y + 1)$ and $\lim_n(G \circ f)(r_n, s_n) = G(x - 1, y - 1)$. It follows that $G \circ f$ cannot be extended to $M \times M$. $\square$

The space $\mathbb{Q}$ is very not Čech-complete. It is natural to wonder how close to Čech-complete a separable and metrizable can be and still satisfy Theorem 3.3.

We can re-use the proof of Theorem 3.3 (and Lemma 3.4) to get an example that is a Baire space.

Example 3.6. We take a subspace A of $\mathbb{R}$ such that $\{x + \pi : x \in A\} = \mathbb{R} \setminus A$. That such a space exists was established by Van Mill in [9] in an alternative proof of Menu's theorem from [8] that $\mathbb{R}$ can be partitioned into two mutually homeomorphic and homogeneous subspaces.

A particularly transparent construction of a set A as required was suggested by Jeroen Bruyning in [9]. Let H be a Hamel base for $\mathbb{R}$ over the field $\mathbb{Q}$ that contains 1 and π.

For $x \in \mathbb{R}$ let x_π denote its (rational) π-coordinate with respect to this base. Let $A = \{x \in \mathbb{R} : \lfloor x_\pi\rfloor \text{ is even}\}$, where $\lfloor\cdot\rfloor$ denotes the greatest-integer function.

By the Baire category theorem the set A is a Baire space. By construction its square $A \times A$ is disjoint from the line L with equation $y = x + \pi$. As before the function $f : A \times A \to [-1, 1]$ defined by

$$f(a, b) = \frac{b - a - \pi}{|b - a - \pi|}$$

has no continuous extension to any point of L.

Finally let G be a G_δ-set that contains A, say $G = \bigcap_{n=1}^{\infty} O_n$ with each O_n open in $\mathbb{R}$. Since A is dense (it contains $\mathbb{Q}$) we find that for every n the set O_n is dense in $\mathbb{R}$ and hence the difference $O_n \setminus A$ is dense in $\mathbb{R} \setminus A$. As $\mathbb{R} \setminus A$ is a Baire space we deduce that $G \setminus A$ is nonempty. Also, $G - \pi$

contains $\mathbb{R} \setminus A$ so that $H = (G \setminus A) \cap (G - \pi) \neq \emptyset$. As above, if $x \in H$ then $(x, x + \pi)$ is in $(G \times G) \cap L$.

Thus the product theorem does not (even) hold for Baire spaces that are separable and metrizable.

There are various properties that are shared by locally compact spaces and completely metrizable spaces, and that imply that the space is a Baire space, see [1]. One can ask for each of these properties whether satisfy our preservation theorems.

Many of these properties have in common that for (separable) metrizable spaces they imply Čech-completeness; this means that counterexamples will have to be non-metrizable.

We show that the property of co-compactness, see also [10], is not strong enough to guarantee that Theorem 2.5 holds.

A space (X, τ) is *co-compact* if there is a family $\mathcal{F}$ of τ-closed sets whose τ-interiors form a base for the given topology τ, and that at the same time forms a subbase for the closed sets of a compact topology $\tau_{\mathcal{F}}$ on the set X. We note that $\tau_{\mathcal{F}}$ will be T_1 but will not necessarily be Hausdorff.

In [10] it is shown that for metrizable spaces co-compactness and Čech-completeness are equivalent.

It turns out that the subject of Gary's first paper [6], the Sorgenfery line $\mathbb{S}$, satisfies Theorem 3.3 too. It is well-known that $\mathbb{S}$ is Lindelöf, and it is readily seen to be co-compact: let $\mathcal{F}$ be the family of closed and bounded intervals in $\mathbb{R}$.

The proof of Theorem 3.3 for $\mathbb{S}$ rests on the following observation about the subset $D = \{\langle x, y\rangle : x + y \geqslant 0\}$ of the plane.

Lemma 3.7. *Let τ be a topology on $\mathbb{R}$ such that D is open in the plane with respect to the product topology from τ. Then for every $a \in \mathbb{R}$ the set $[a, \infty)$ belongs to τ, and hence τ is not second-countable.*

Proof. Let $x \in \mathbb{R}$ and let U and V be members of τ such that $\langle x, -x\rangle \in U \times V \subseteq D$. We claim that $U \subseteq [x, \infty)$ (and by symmetry $V \subseteq [-x, \infty)$). Indeed, let $z \in U$, then $\langle z, -x\rangle \in U \times V \subseteq D$ and hence $z - x \geqslant 0$, or $z \geqslant x$.

Let $\langle U_x : x \in \mathbb{R}\rangle$ be a choice function from τ such that $x \in U_x \subseteq [x, \infty)$ for all x. Then $[a, \infty) = \bigcup_{x \geqslant a} U_x$ is in τ, for every a.

That τ is not second-countable follows as in the familiar proof that the Sorgenfrey line is not second-countable. $\square$

Proof of Theorem 3.3 for $\mathbb{S}$. Let $f : \mathbb{S} \times \mathbb{S} \to \mathbb{R}$ be the characteristic function of D and assume $F : \mathbb{R}^{C(\mathbb{S})} \times \mathbb{R}^{C(\mathbb{S})} \to \mathbb{R}$ is a continuous extension of F. We take a countable subfamily E of $C(\mathbb{S})$ and a continuous function

$G : \mathbb{R}^E \times \mathbb{R}^E \to \mathbb{R}$ such that $F = G \circ (\pi_E \times \pi_E)$. If we make sure that the identity function $i : \mathbb{S} \to \mathbb{R}$ belongs to E then π_E is injective on $\mathbb{S}$ and G restricts to the characteristic function of D on $\mathbb{S} \times \mathbb{S}$.

It follows that D is open in the topology τ that $\mathbb{S} \times \mathbb{S}$ inherits from $\mathbb{R}^E \times \mathbb{R}^E$, but the subspace $(\mathbb{S}, \tau)$ is second-countable. $\square$

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