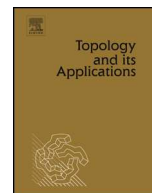




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ABSTRACT

This article attempts to survey Alan Dow's contributions to General and Set-theoretic Topology.

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Alan Dow celebrated his 60th birthday on the 5th of December 2014 by flying to Ithaca, NY, to undergo a conference in honour of that birthday. This article is an attempt to present an overview of Alan's many contributions to General and Set-theoretic Topology.

1. Elementarity

One of the many gifts from Alan to Set-theoretic Topology is the use of elementarity. For a while this was even known as “Dow’s method of elementary submodels”. But Alan would be, was, and still is the first to protest that the Löwenheim–Skolem theorem predates him by a few decades.

We have for the longest time been familiar with recursive constructions where often beforehand a sequence of situations/sets is set up and during the construction witnesses to bad things will be eliminated or witnesses to good things will be embraced. In the end we consider such a situation and realize that it was basically dealt with during the construction. A very good example is the Pol–Shapirovskiĭ proof of Arhangel’skiĭ’s theorem on the cardinality of compact first-countable spaces.

What set theorists realized was that one can reduce the length of such proofs considerably by an application of the Löwenheim–Skolem theorem to a model of ‘enough set theory’: its proof is the ultimate closing-off argument where one deals with *all possible* situations in one go (even ones that will never occur in your problem at hand). But, and this is where this method gets its power, you will certainly have dealt with every eventuality related to your problem. Basically what is left is to perform what would have been the final step of your old recursive argument. This requires some familiarity with first-order logic and model theory, so that you know how far you can go with your arguments. But the time spent learning that will pay itself back handsomely in time saved later.

A good place to start learning this is Alan’s first introduction, [7], which has an elementary proof of Arhangel’skiĭ’s theorem that one should put next to the Pol–Shapirovskiĭ argument to see the difference between the ‘standard’ and the elementarity mindset. A later survey, [14], gives more applications of the latter.

2. Remote points

If X is completely regular then it is dense in βX , so every point of X^* lies close to X ; however some points lie closer to X than others. One can formulate degrees of closeness by stipulating that the point belongs to the closure of a topologically small subset of X . Thus, for example, $p \in X^*$ is *near* if $p \in \text{cl}_\beta D$ for some closed and discrete subset of X ; other variants can be obtained by using relatively discrete subsets, scattered subsets, and nowhere dense sets. The negation of the last notion has proved to be very fruitful: call $p \in X^*$ a *remote point* of X if $p \notin \text{cl}_\beta A$ for all nowhere dense subsets of X . Remote points were introduced by Fine and Gillman in 1962 who proved that the Continuum Hypothesis (abbreviated CH) implies that the real line \mathbb{R} has a remote point. Actually, their proof applies to every separable and non-pseudocompact space. Around 1980 van Douwen, and independently Chae and Smith, proved in ZFC that every non-pseudocompact space with countable π -weight has remote points. That there are spaces without remote points, was demonstrated by van Douwen and van Mill. In 1982, Alan took over the research on remote points completely, leaving absolutely nothing for his competitors (see [1–3, 5, 6, 8, 20, 28, 36]). He became the world’s expert on remote points. He substantially improved the results of van Douwen, and Chae and Smith by showing that every non-pseudocompact ccc space of π -weight at most ω_1 has a remote point, and that under CH the bound ω_1 is not optimal. The fruits of remote points are manifold. The points themselves were used in ‘honest’ proofs of non-homogeneity of certain Čech–Stone remainders: for example \mathbb{Q}^* is extremally disconnected at each remote point but not at other points. The techniques developed and used for their construction have found many applications too. Alan’s proof that $\omega \times 2^\kappa$ has remote points gave new insight in the structure of the partial order that adds Cohen reals: a remote point, seen as a clopen

filter on $\omega \times 2^\kappa$, takes big bites out of dense open sets and these bites combine to form approximations of generic filters, called *enDowments* by some. These enDowments were crucial in a Cohen-real proof of the consistency of the normal Moore space conjecture, [10,11]. We must also mention that Alan showed that the result of Fine and Gillman needs extra assumptions: in the side-by-side Sacks model there is a non-pseudocompact separable space without remote points, [8].

3. $\beta\mathbb{N}$ and $\beta\mathbb{R}$

The Čech–Stone compactification $\beta\mathbb{N}$ of the discrete space of natural numbers, \mathbb{N} , is Alan’s favorite space, just like the space of rational numbers, \mathbb{Q} , was Eric van Douwen’s favorite space. Alan’s work concentrates on the construction of special ultrafilters on \mathbb{N} such as weak P -points, cozero-accessible points, (bow)tie-points, certain (finite-to-one) maps defined on $\beta\mathbb{N}$ or $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$, and closed subsets of \mathbb{N}^* . Although Alan can be thought of as a set theorist, his thinking is topological: he is interested in those set theoretical aspects of $\beta\mathbb{N}$ that shed light on topological questions.

He demonstrated his $\beta\mathbb{N}$ -talents already quite early in his career. In [13] he gave an overview of certain aspects of $\beta\mathbb{N}$ that is still valuable today. His proof presented there that there are $2^{\mathfrak{c}}$ Rudin–Keisler incomparable ultrafilters on \mathbb{N} , is the best one around.

The space $\beta\mathbb{N}$ contains many copies of itself. If D is any countable discrete subset of $\beta\mathbb{N}$, then its closure is a (topological) copy of $\beta\mathbb{N}$. Hence the space \mathbb{N}^* contains many topological copies of itself as well. This prompted Eric van Douwen to ask whether there is a copy of \mathbb{N}^* in \mathbb{N}^* that is not of the form $\text{cl } D \setminus D$ for some countable discrete subset D of \mathbb{N}^* . Such a copy of \mathbb{N}^* is called *non-trivial*. Under various assumptions, non-trivial copies of \mathbb{N}^* exist. But what about such copies in ZFC? This longstanding open problem, which was thought to be beyond reach by experts, was solved in the affirmative by Alan in [34].

In [17], he showed with Hart that the Open Coloring Axiom implies that the Stone space of the measure algebra is not a continuous image of \mathbb{N}^* , which contradicts an earlier result of Frankiewicz and Gutek. Alan’s work related to van Douwen spaces [22,24–26] gives answers to very natural problems on spaces that are continuous images of $\beta\mathbb{N}$ or \mathbb{N}^* under finite-to-one mappings.

One of Alan’s favorite results about $\beta\mathbb{N}$ is the existence of a tree π -base for \mathbb{N}^* , so naturally he would investigate the possible structure of these, [9], as well as employ them in the study of the absolute of \mathbb{N}^* , [4,16]. The latter paper ties in with another Čech–Stone compactification that has been the beneficiary of Alan’s interest: $\beta\mathbb{R}$. As with $\beta\mathbb{N}$, it is mostly \mathbb{R}^* , or rather one of its halves, \mathbb{H}^* , whose structure we would like to clear up — $\mathbb{H} = [0, \infty)$.

One thing to do is look for parallels between \mathbb{N}^* and \mathbb{H}^* : take a known result on \mathbb{N}^* and reformulate it to take into account that \mathbb{H}^* is a continuum. Sometimes this works, as in [18]: there is a complete parallel version of Parovičenko’s theorem: every continuum of weight \aleph_1 or less is a continuous image of \mathbb{H}^* and thus, under CH, the continua of weight \mathfrak{c} or less are exactly the continuous images of \mathbb{H}^* . The parallel also extends in the negative direction: many examples of non-images of \mathbb{N}^* have a connected counterpart.

Sometimes the parallel breaks down: every separable compact space is clearly a continuous image of \mathbb{N}^* but there is a separable continuum that, consistently, is not a continuous image of \mathbb{H}^* , see [27]. One parallel is still unresolved: is every perfectly normal continuum a continuous image of \mathbb{H}^* ?

For the longest time we knew of only a few (i.e., finitely many) distinct subcontinua of \mathbb{H}^* ; just recently that number was raised to the maximum possible $2^{\mathfrak{c}}$. If CH *does not* hold then this follows from results on ultrapowers of linear orders, [30], the construction of such a family in case CH does hold is more involved, [35].

4. Convergence

Another recurring theme in Alan’s work is that of convergence, in particular as related to the closure operation. We all know that in a first-countable space a point belongs to the closure of a set if and only

if there is a sequence in that set that converges to the point. If one puts universal quantifiers in front of ‘point’ and ‘set’ then one obtains the definition of Fréchet–Urysohn spaces, if one allows for well-ordered sequences of arbitrary length then one defines radial spaces. A weakening of the Fréchet–Urysohn property, sequentiality, states that a set is closed iff it contains all limits of converging sequences whose terms belong to the set, allowing for well-ordered sequences of arbitrary length will define pseudo-radial spaces. Finally, a space is countably tight (or has countable tightness) if $\text{cl } A = \bigcup \{\text{cl } B : B \in [A]^{\leq \aleph_0}\}$ holds for every subset A of the space.

The question whether compact spaces of countable tightness are sequential is known as the Moore–Mrówka Problem; after Balogh showed that PFA implies a yes answer Alan established its outright consistency and more [12]; later he showed that it is even consistent that compact spaces of countable tightness and weight \aleph_1 are Fréchet–Urysohn [15].

The sequential closure $\text{cl}_s A$ of a set A in a topological space is equal to the set of limits of all converging sequences with terms in A (constant sequences ensure $A \subseteq \text{cl}_s A$). In a Fréchet–Urysohn space we have $\text{cl } A = \text{cl}_s A$ for all subsets; in a sequential space one obtains $\text{cl } A$ after iterating cl_s at most ω_1 many times — the sequential order of a space is the minimum ordinal α such that all closures are reached in at most α steps. Under CH there are compact sequential spaces of all possible sequential orders; without CH much less is known: in [21] Alan constructed an example from $\mathfrak{b} = \mathfrak{c}$ of compact space of scattered height and sequential order 5 and later he showed, under PFA, that the known constructions would not yield spaces of order higher than ω [31].

Efimov’s problem may also be put under the heading of convergence: it asks whether every compact Hausdorff space contains a non-trivial converging sequence or a copy of $\beta\mathbb{N}$. Fedorchuk’s compact S-space from \diamond is a particularly strong counterexample: hereditarily separable and without converging sequences. The consistency of a positive answer seems to be getting a more and more remote possibility, not least because of examples that Alan (co-)constructed: there is one if $\text{cf}([\mathfrak{s}]^{\aleph_0}, \subset) = \mathfrak{s}$ and $2^{\mathfrak{s}} < 2^{\mathfrak{c}}$ hold [23] and also one if $\mathfrak{b} = \mathfrak{c}$ [32] (with Shelah).

The latter example helps settle another problem on convergence: does every compact Hausdorff space contain a converging ω -sequence or a converging ω_1 -sequence (Hušek); as $\beta\mathbb{N}$ contains a sequence of the latter type, this question can be seen as a weakening of Efimov’s problem. That it really is a weakening follows from results in [33]: in various models, including the standard model for $\mathfrak{b} = \mathfrak{c}$, compact spaces without converging ω_1 -sequences are first-countable.

5. MAD families

A look at some of the papers cited above will show that one of Alan’s tools of choice is a Maximal Almost Disjoint (MAD) family. They are also objects of study in their own right. A family, \mathcal{A} , of countably infinite subsets of a set X is almost disjoint if any two distinct elements have a finite intersection and a MAD family is maximal among such families with respect to inclusion. One topologizes $X \cup \mathcal{A}$ by making the points of X isolated and by letting the basic neighborhoods of $A \in \mathcal{A}$ be the sets of the form $\{A\} \cup A \setminus F$, where F is finite. The resulting space, denoted $\psi(X, \mathcal{A})$, can have many striking properties. Alan’s interest has been in the Čech–Stone remainders of such spaces, especially in the case \mathcal{A} is maximal and $X = \mathbb{N}$. Mrówka showed that this remainder can be a singleton, and Alan and Jerry Vaughan investigated what happens for uncountable X in [29]. One of the results that gave Alan great satisfaction deals with representable algebras, or their duals, representable spaces: these are the zero-dimensional compact spaces that can occur as $\psi(\mathbb{N}, \mathcal{A})^*$ for some MAD family \mathcal{A} . Baumgartner and Weese initiated the study of these and proved that, as one is wont to expect, all is well under the assumption of CH: one gets the compact zero-dimensional spaces of weight \mathfrak{c} or less. In [19], in a veritable tour de force, Alan and his student JinYuan Zhou presented a consistent example of a mad family \mathcal{D} with two points, a and b , in $\psi(\mathbb{N}, \mathcal{D})^*$ such that the quotient of

the remainder obtained by identifying just the points a and b is not of the form $\psi(\mathbb{N}, \mathcal{A})^*$ for any MAD family \mathcal{A} .

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