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Chapter 9

The Čech-Stone compactification of the Real line

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Introduction

The Čech-Stone compactification of the real line $\mathbb{R}$ seems to have been much less investigated than the space $\beta\omega$. One reason for this is that $\beta\omega$ can be attacked with two kinds of weapons: Boolean algebraic and topological. The space $\beta\mathbb{R}$ is a continuum, so it is not susceptible to Boolean algebraic treatment. Be that as it may, $\beta\mathbb{R}$ deserves more of our attention than it has received so far. In an effort to catch this attention I have tried to give a coherent overview of our knowledge of $\beta\mathbb{R}$ at this time.

There have been several interesting developments in $\beta\mathbb{R}$ recently. As an answer to a well-known question of van Douwen, Yu [1991] constructed an autohomeomorphism of $\mathbb{R}^*$ that maps a remote point to a non-remote point. Another recent result is actually about ω^*; it is well-known that the principle NCF implies that ω^* can be covered by nowhere dense P-sets; in [1991a] Zhu has shown that NCF in fact implies that ω^* can be covered by a *chain* of nowhere dense P-sets. The proof makes essential use of the structure of the family of subcontinua of $\mathbb{R}^*$.

Acknowledgements. I would like to thank several people for providing me with material to write about.

Jan van Mill kindly gave me access to the unpublished work of Eric van Douwen (hereafter cited as 'van Douwen's notes')

Jiang-Ping Zhu sent me his preprints on continua in $\mathbb{R}^*$; as the reader will see, many results of van Douwen were rediscovered by him, mostly with different proofs.

Michel Smith provided me with a preprint of Joseph Yu's work on autohomeomorphisms of $\mathbb{R}^*$.

1. Notation and conventions

Although as a rule we introduce a notion when we first need it there are some things that should be said right at the beginning.

The Čech-Stone compactification probably needs no introduction any more but we fix our notation anyway. The properties of βX that we use are (i) every continuous function from X to the unit interval $\mathbb{I}$ has a continuous extension over βX and (ii) if A and B are closed subsets of X then $\mathrm{cl}_{\beta X} A \cap \mathrm{cl}_{\beta X} B = \mathrm{cl}_{\beta X}(A \cap B)$.

Property (ii) is valid only for normal spaces but since we are dealing with subspaces of $\mathbb{R}$ there will arise no problem with this. We shall denote the space $\beta X \setminus X$ invariably by X^* and we call it the (Čech-Stone) remainder of X.

We shall freely identify the points of βX with maximal filters of closed sets; if $x \in \beta X$ then x corresponds to

$$\mathcal{F}_x = \{F \subseteq X : F \text{ is closed in } X \text{ and } x \in \mathrm{cl}_{\beta X} F\}.$$

That $\mathcal{F}_x$ is a filter follows from property (ii) above. Thus, a base for a point of βX is actually a base for the filter $\mathcal{F}_x$. We trust that the reader will see through this usage.

The spaces that we deal with — $\mathbb{H}$ and $\mathbb{M}$ ($\mathbb{M}$ is defined below in subsection 2.1) — are σ-compact and locally compact. This has several useful consequences for their remainders: if X is locally compact then X^* is compact and if X is in addition σ-compact then X^* is an F-space in which every nonempty G_δ-set has nonempty interior. The characteristic property of F-spaces is that bounded continuous functions defined on F_σ-subsets can be extended over the whole space. This means for us that if A is an F_σ-subset of X^* then $\overline{A} = \beta A$. We shall use this fact a few times in this article (Proposition 2.12, Theorem 6.5 and the construction of the continuum K_8 in Section 10).

The book GILLMAN and JERISON [1976] is one of the basic references for βX.

1.1. Some notation

We need to agree on a minimum of notation beforehand. The closure of a set A is usually denoted $\mathrm{cl}_X A$, where X is the ambient space. Sometimes it will be convenient to use just a bar over the set in question. We expect no confusion.

If U is an open set in a space X then $\mathrm{Ex}\, U$ will be the largest open subset of βX whose intersection with X equals U: in formula $\mathrm{Ex}\, U = \beta X \setminus \overline{X \setminus U}$.

Finally there is the space that it is all about: the half line $\mathbb{H} = [0, \infty)$. We use $\mathbb{H}$ rather than $\mathbb{R}$ because $\mathbb{R}^*$ is merely the topological sum of two copies of $\mathbb{H}^*$.

1.2. Continua

A continuum is a compact and connected Hausdorff space. Again, most of the notions will be defined when their time comes but we should define here one of the central notions in our investigations. A point x of a continuum X is said to be a *cut point* if the space $X \setminus \{x\}$ is not connected. This is equivalent to saying that there are two non-degenerate closed sets F and G such that $F \cap G = \{x\}$.

The reference KURATOWSKI [1968, Chapter Five] still is the best source for basic material on continua.

1.3. Set Theory

We use no heavy set-theoretic machinery. On two occasions we employ the forcing method but not to such an extent that we should explain the notation and terminology involved.

The book KUNEN [1980] contains all the set theory that we need.

2. Sums of continua

In this section we discuss a general way of constructing continua in Čech-Stone remainders. We start with the following elementary topological lemma. Recall that a map between topological spaces is said to be *monotone* if every fiber of it is connected.

2.1. LEMMA. *Let $f : X \to Y$ be a perfect and monotone map. Then the map $\beta f : \beta X \to \beta Y$ is also monotone.*

□ It suffices to show that $\beta f^{\leftarrow}(z)$ is connected for every $z \in Y^*$. So let $z \in Y^*$ and write $\beta f^{\leftarrow}(z)$ as the disjoint union of two closed sets A and B. Using normality of βX find open sets U and V around A and B respectively, whose closures are disjoint. As the open set $U \cup V$ contains $\beta^{\leftarrow}(z)$ there is an open set O in βY containing z such that $\beta f^{\leftarrow}[O] \subseteq U \cup V$; after shrinking U and V a bit we may as well assume that $\beta f^{\leftarrow}[O] = U \cup V$.

The set $(U \cup V) \cap X$ is saturated with respect to f: it equals $f^{\leftarrow}[O \cap Y]$. Since $f^{\leftarrow}(y)$ is connected for every $y \in Y$ and since U and V are disjoint open sets in βX we see that $f^{\leftarrow}(y)$ is contained in U or V for every $y \in O$. Therefore $U \cap X$ and $V \cap X$ are saturated with respect to f as well. We conclude that $U' = f[U \cap X]$ and $V' = f[V \cap X]$ are disjoint open sets in Y; moreover $O \cap Y = U' \cup V'$.

We claim that $\overline{U'} \cap \overline{V'} \cap O = \varnothing$. To see this consider $x \in O$ and assume $x \in \overline{U'}$. Let $g : \beta Y \to \mathbb{I}$ be continuous such that $g(x) = 1$ and $g[\beta Y \setminus O] \subseteq \{0\}$. Using g we define $h : Y \to [-1, 1]$ by

$$h(y) = \begin{cases} g(y) & \text{if } y \in Y \setminus V', \text{ and} \\ -g(y) & \text{if } y \in Y \setminus U'. \end{cases}$$

Observe that h is continuous and that $|h| = g \restriction Y$, hence $|\beta h| = g$. Now $h(y) = g(y) \geq 0$ for $y \in U'$ so that $\beta h(x) = g(x) = 1$; on the other hand $h(y) = -g(y)$ for $y \in V'$, hence $\beta h(y) \leq 0$ for all $y \in \overline{V'}$. We see that $x \notin \overline{V'}$.

Assume for example that $z \notin \overline{V'}$. Now $V \subseteq \overline{V \cap X} \subseteq \beta f^{\leftarrow}[\overline{V'}]$, because $V \cap X = f^{\leftarrow}[V']$. But then $B \subseteq V \cap \beta f^{\leftarrow}(z) = \varnothing$. It follows that $\beta f^{\leftarrow}(z)$ is connected. □

We use this lemma to produce many continua in Čech-Stone remainders. First we introduce some notation. Let $X = \bigoplus_{n \in \omega} K_n$ be a topological sum of continua. The map from X to ω that sends K_n to n will indiscriminately be denoted by π. The following corollary to Lemma 2.1 identifies the components of βX.

2.2. COROLLARY. *The components of βX are the fibers of the map $\beta\pi$.*

Thus, for every point u of $\beta\omega$ the fiber $\beta\pi^{\leftarrow}(u)$ is a continuum, we usually denote it by K_u. Note that

$$K_u = \bigcap_{U \in u} \mathrm{cl}_{\beta X} \bigcup_{n \in U} K_n.$$

We extend the K_u notation to cover sequences $\langle F_n \rangle_n$ of closed sets of X that satisfy $F_n \subseteq K_n$ for every n. In this case we let $F_u = \bigcap_{U \in u} \mathrm{cl}_{\beta X} \bigcup_{n \in U} F_n$.

If $\langle x_n \rangle_n$ is a sequence of points in X such that $x_n \in K_n$ for all n then x_u denotes the point $u{-}\lim_n x_n$ (note that x_u is the unique accumulation point of $\langle x_n \rangle_n$ in K_u). The following lemma provides us with cut points in the continua K_u.

2.3. LEMMA. *Let $\langle x_n \rangle_n$ be a sequence of points in X such that $x_n \in K_n$ for all n and let $u \in \omega^*$. Then x_u is a cut point of K_u if and only if the set $\{n : x_n \text{ is a cut point of } K_n\}$ belongs to u.*

□ One direction is easy: if, without loss of generality, for every n one can write $K_n = F_n \cup G_n$ where F_n and G_n are non-trivial closed sets with intersection $\{x_n\}$ then $K_u = F_u \cup G_u$ and $F_u \cap G_u = \{x_u\}$.

For the other direction let U and V be open sets in βX that both intersect K_u such that $U \cap V \cap K_u = \varnothing$ and $K_u \setminus (U \cup V) = \{x_u\}$. It follows immediately that the set

$$A = \{n : U \cap K_n \neq \varnothing \neq V \cap K_n \text{ and } x_n \notin U \cup V\}$$

belongs to u. A moment's reflection should show that $\bigl\{n \in A : \{x_n\} = K_n \setminus (U \cup V)\bigr\}$ also belongs to u. □

2.1. The space $\mathbb{M}$

We now consider a special case, one that will be very important in the study of $\mathbb{H}^*$. We define $\mathbb{M} = \omega \times \mathbb{I}$. We shall write $\mathbb{I}_n$ for $\{n\} \times \mathbb{I}$ and $\mathbb{I}_u$ for the fiber of u. To add to the confusion we also write $x_u = u\text{-}\lim_n \langle n, x_n\rangle$ if $\langle x_n\rangle_n$ is a sequence in $\mathbb{I}$. In addition we put $0_u = u\text{-}\lim_n 0$ and $1_u = u\text{-}\lim_n 1$.

The reason for our interest in $\mathbb{M}$ is that it provides a lot of subcontinua of $\mathbb{H}^*$. For consider an embedding e of $\mathbb{M}$ into $\mathbb{H}$ such that $\lim_n e(n,0) = \infty$. Then βe embeds $\beta\mathbb{M}$ into $\beta\mathbb{H}$ and $\mathbb{M}^*$ into $\mathbb{H}^*$ so that we get a copy of $\mathbb{I}_u$ in $\mathbb{H}^*$ for every u in ω^*. We shall see later that these continua determine virtually the whole structure of the whole family of subcontinua of $\mathbb{H}^*$. We shall call them *standard subcontinua* of $\mathbb{H}^*$.

Another way of embedding the components of $\mathbb{M}^*$ into $\mathbb{H}^*$ uses a very simple quotient map of $\mathbb{M}^*$ onto $\mathbb{H}^*$. The *shift on* ω is the map σ defined by $\sigma(n) = n+1$; we denote its extension to $\beta\omega$ by σ as well.

2.4. THEOREM. *If in $\mathbb{M}^*$ one identifies, for every $u \in \omega^*$, the point 1_u with $0_{\sigma(u)}$ then the quotient space is homeomorphic with $\mathbb{H}^*$.*

□ The proof is easy once one realizes that $\mathbb{H}$ is the quotient of $\mathbb{M}$ that one obtains by identifying $\langle n, 1\rangle$ with $\langle n+1, 0\rangle$ for every n and that the restriction to $\mathbb{H}^*$ of the Čech-Stone extension of this quotient map is exactly the map from the theorem. □

This quotient map embeds every $\mathbb{I}_u$ into $\mathbb{H}^*$ because, as is easily seen, it is one-to-one on each $\mathbb{I}_u$.

2.2. Properties of the continua $\mathbb{I}_u$

Let us fix a point u of ω^*. We shall determine some elementary properties of $\mathbb{I}_u$. The following theorem identifies the more obvious cut points of $\mathbb{I}_u$. It is a direct consequence of Lemma 2.3.

2.5. THEOREM. *If $\langle x_n\rangle_n$ is a sequence in $(0,1)$ then the point x_u is a cut point of $\mathbb{I}_u$.*

We use P_u to denote the set of all points x_u for sequences $\langle x_n\rangle_n$ in $\mathbb{I}$. This set admits a natural linear order:

$$x_u <_u y_u \quad \text{iff} \quad \{n : x_n < y_n\} \in u.$$

The following proposition summarizes the relevant information about P_u.

2.6. Proposition. *The set $P_u \setminus \{0_u, 1_u\}$ is a dense set of cut points of $\mathbb{I}_u$ and its subspace topology is the same as the order topology induced by $<_u$.*

□ That every point of P_u is a cut point of $\mathbb{I}_u$ is the content of Theorem 2.5. To show that P_u is dense we let U be an open subset of $\beta\mathbb{M}$ such that $U \cap \mathbb{I}_u \neq \varnothing$. It follows that $A = \{n : U \cap \mathbb{I}_n \neq \varnothing\}$ is an element of u. For every $n \in A$ choose an interval (a_n, b_n) in $\mathbb{I}_n$ that is contained in U and let $t_n = (a_n + b_n)/2$. Then $t_u \in U \cap P_u$.

Observe that if $t_u \in U \cap P_u$ would have been given in advance we could have picked a_n and b_n in such a way that $t_n \in (a_n, b_n) \subseteq U$. This then shows that $t_u \in (a_u, b_u) \subseteq U \cap P_u$ so that the subspace topology on P_u is contained in the order topology.

For the reverse inclusion simply note that the interval $[0_u, x_u)$ is exactly $P_u \setminus G_u$ and hence open in the subspace topology, where G_u has the established meaning once we set $G_n = \{n\} \times [x_n, 1]$ for all n. The interval $(x_u, 1_u]$ is likewise open. □

This proposition and Theorem 2.5 give rise to the following definition.

2.7. Definition. Let a_u and b_u be points of P_u with $a_u <_u b_u$. The *interval from a_u to b_u in $\mathbb{I}_u$*, denoted $[a_u, b_u]$, is the set of those points of $\mathbb{I}_u$ that are in the closure of $\bigcup_n [a_n, b_n]$.

If x is any point of $\mathbb{I}_u$ then we define the *layer* of x to be the intersection of all intervals that contain x. We denote this layer by L_x.

We collect some useful properties of layers and intervals.

2.8. Proposition. *Let a_u and b_u be points of P_u with $a_u <_u b_u$ and let x be a point of $\mathbb{I}_u$.*

1. *The interval $[a_u, b_u]$ is homeomorphic with $\mathbb{I}_u$.*
2. *The interval $[a_u, b_u]$ is irreducible between a_u and b_u.*
3. *$[a_u, b_u] = [0_u, b_u] \cap [a_u, 1_u]$; hence L_x is the intersection of all intervals of the form $[0_u, b_u]$ or $[a_u, 1_u]$ that contain it.*
4. *The layers of the points in P_u are one-point sets.*

It follows that each layer is a continuum: it is the intersection of a directed family of continua.

Using Proposition 2.8 we extend the order $<_u$ to the whole set of layers in the following way: $L_x <_u L_y$ iff there is a point $a_u \in P_u$ such that $L_x \subseteq [0_u, a_u]$ and $L_y \subseteq [a_u, 1_u]$. This is equivalent to saying that x and y have elements F and G respectively such that $\{n : F \cap \mathbb{I}_n < G \cap \mathbb{I}_n\}$ belongs to u. Notations like $[0_u, L_x)$ will have the obvious meaning (in this case it is the union of the set of layers below L_x).

The following lemma establishes an important continuity property of the extended ordering $<_u$.

2.9. Lemma. *Let $x \in \mathbb{I}_u \setminus \{0_u, 1_u\}$. The closure of the interval $[0_u, L_x)$ is the interval $[0_u, L_x]$ $(= [0_u, L_x) \cup L_x)$.*

□ If $x \in P_u$ then this follows from the fact that P_u is dense in $\mathbb{I}_u$ and that its subspace topology is its order topology.

Assume $x \notin P_u$. A moment's reflection should convince us that it suffices to show $x \in \overline{[0_u, L_x)}$. Let V be an open neighbourhood of x in $\beta\mathbb{M}$ and let $A = \{n : V \cap \mathbb{I}_n \neq \varnothing\}$. The set A is in u. For $n \in A$ let $a_n = \inf\{t : \langle n, t\rangle \in V\}$ and $b_n = \sup\{t : \langle n, t\rangle \in V\}$. Consider the interval $[a_u, b_u]$; it clearly contains x, hence the whole set L_x. Since $x \notin P_u$ we can find $d_u \in P_u$ with $a_u <_u d_u <_u L_x$. We may assume that $a_n < d_n$ for $n \in A$; therefore we can choose, for $n \in A$, a point $t_n \in (a_n, d_n)$ such that $\langle n, t_n\rangle \in V$. But then $t_u \in \overline{V} \cap [0_u, L_x)$. This suffices by regularity of the space $\beta\mathbb{M}$. □

It goes (almost) without saying that a similar formula holds for the closure of $(L_x, 1_u]$. We also note that the intervals $[0_u, L_x)$ and $(L_x, 1_u]$ are connected since each is the union of an increasing chain of intervals.

2.10. Corollary. *The decomposition of $\mathbb{I}_u$ into layers is upper semicontinuous and the quotient topology is exactly the order topology from the ordering $<_u$.*

We denote the quotient space obtained in this way by X_u; it is readily seen that X_u is the Dedekind completion of P_u. For later use we give a description of those subcontinua of $\mathbb{I}_u$ that meet at least two layers.

2.11. Theorem. *Let K be a subcontinuum of $\mathbb{I}_u$ that meets two different layers. Then there are two points x and y of $\mathbb{I}_u$ with $L_x <_u L_y$ such that $K = [L_x, L_y]$. Moreover, K is irreducible between p and q whenever $p \in L_x$ and $q \in L_y$.*

□ Consider the quotient map $q : \mathbb{I}_u \to X_u$. The image $q[K]$ is connected and contains, by assumption, at least two points. It follows that $q[K]$ is a non-degenerate interval of X_u, say with endpoints L_x and L_y. We claim that $K = [L_x, L_y]$ in $\mathbb{I}_u$. Indeed, since $q \restriction P_u$ is one-to-one we must have $[L_x, L_y] \cap P_u \subseteq K$. As $\overline{[L_x, L_y] \cap P_u} = [L_x, L_y]$ and also $K \subseteq [L_x, L_y]$ we get $K = [L_x, L_y]$.

The irreducibility follows easily: any continuum meeting L_x and L_y must contain $[L_x, L_y] \cap P_u$. □

To finish this section let us prove that $\mathbb{I}_u$ contains non-trivial layers.

2.12. Proposition. *Let $\langle a_n\rangle_n$ be a strictly increasing sequence in P_u and let $B = \{b \in P_u : a_n <_u b$ for all $n\}$. Then*

$$L = \bigcap\bigl\{[a_n, b] : n \in \omega, b \in B\bigr\}$$

is a non-trivial layer of $\mathbb{I}_u$.

□ That L is a layer follows from the fact that P_u is dense in the ordering $<_u$. Since the set $D = \{a_n : n \in \omega\}$ is relatively discrete its closure is homeomorphic to $\beta\omega$, but $\overline{D} \setminus D \subseteq L$. □

2.13. Remark. Layers may also be defined in a purely topological way: first we give an alternative definition of the ordering $<_u$: we say $x \leq_u y$ iff every continuum that contains 0_u and y also contains x. The layer of x is now the set of those points y for which $x \leq_u y$ and $y \leq_u x$. It is not overly difficult to show that we get the same layers back.

This clearly demonstrates that layers are topologically invariant: any homeomorphism between continua $\mathbb{I}_u$ and $\mathbb{I}_v$ must map layers to layers.

Notes for Section 2.

Lemma 2.1 is probably folklore, but I did not find it in the literature. It appears in van Douwens's notes.

The space $\beta\mathbb{M}$ and the continua $\mathbb{I}_u$ were studied extensively by Mioduszewski in [1978]. The results of Subsections 2.1 and 2.2 are taken from that paper.

The definition of layers as given in Remark 2.13 was carried out for metric irreducible continua in Kuratowski [1968, p. 199].

3. A nice base for $\beta\mathbb{H}$

Time and time again we shall need nice open sets in $\beta\mathbb{H}$ and $\mathbb{H}^*$. In this section we shall describe such sets and show that there are indeed enough of them.

Let U and V be open subsets of $\beta\mathbb{H}$ that both meet $\mathbb{H}^*$, whose closures are disjoint and assume that $\inf U < \inf V$. We shall define, inductively, two sequences $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ in $\mathbb{H}$ and use these to construct nice open sets around U and V.

Let $a_0 = \inf U$. If n is even we let $b_n = \sup\{x \in U : (a_0, x) \cap V = \varnothing\}$ and $a_{n+1} = \inf\{x \in V : x > b_n\}$. If n is odd we reverse the roles of U and V.

We note that $a_n < b_n < a_{n+1}$ for every n because the closures of U and V are disjoint. Also, because U and V both meet $\mathbb{H}^*$, the construction will never stop and both sequences will converge to infinity.

Now let $U_1 = \bigcup_n (a_{2n}, b_{2n})$ and $V_1 = \bigcup_n (a_{2n+1}, b_{2n+1})$. We consider the sets $\operatorname{Ex} U_1$ and $\operatorname{Ex} V_1$. Since $U \cap \mathbb{H} \subseteq U_1$ and $V \cap \mathbb{H} \subseteq V_1$ we know that $U \subseteq \operatorname{Ex} U_1$ and $V \subseteq \operatorname{Ex} V_1$. Furthermore the closures—in $\mathbb{H}$—of U_1 and V_1 are disjoint, hence so are the closures—in $\beta\mathbb{H}$—of $\operatorname{Ex} U_1$ and $\operatorname{Ex} V_1$.

Let us call open sets like $\operatorname{Ex} U_1$ and $\operatorname{Ex} V_1$ that come from discrete sequences of open intervals *standard open sets*. We summarize the foregoing discussion in the following lemma.

3.1. Lemma. *If U and V are open subsets of $\beta\mathbb{H}$ that both meet $\mathbb{H}^*$ and whose closures are disjoint then they can be separated by standard open sets whose closures are disjoint as well.*

We shall often use the following consequence of this lemma.

3.2. Proposition. *If F and G are disjoint closed subsets of $\mathbb{H}^*$ then they can be separated by standard open sets with disjoint closures. In particular, if U is an open subset of $\mathbb{H}^*$ containing F then there is a standard open set O such that $F \subseteq O \subseteq U$.*

4. $\mathbb{H}^*$ is indecomposable

We begin by recalling the definition of indecomposable continua.

4.1. DEFINITION. A continuum is said to be *indecomposable* if it can not be written as the union of two proper subcontinua.

It is an instructive exercise to show that a continuum is indecomposable iff every proper subcontinuum of it is nowhere dense. We shall mostly use indecomposability in this form.

4.2. THEOREM. *The space $\mathbb{H}^*$ is an indecomposable continuum.*

□ We must show that every proper subcontinuum of $\mathbb{H}^*$ is nowhere dense.

Let K be a proper subcontinuum of $\mathbb{H}^*$ and take a point x in $\mathbb{H}^* \setminus K$. Apply Proposition 3.2 to K and x to obtain sequences $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ such that $a_n < b_n < a_{n+1}$ for all n and $K \subseteq \operatorname{Ex} O$, where $O = \bigcup_n (a_n, b_n)$.

Consider the following collection of subsets of ω:

$$u = \bigl\{ A \subseteq \omega : K \subseteq \operatorname{cl}_{\beta\mathbb{H}} F(a,b,A) \bigr\},$$

where $F(a,b,A)$ denotes the set $\bigcup_{n\in A}[a_n, b_n]$. From the fundamental property $\operatorname{cl}_{\beta\mathbb{H}}(F \cap G) = \operatorname{cl}_{\beta\mathbb{H}} F \cap \operatorname{cl}_{\beta\mathbb{H}} G$ we deduce that u is a filter. In fact, because K is connected, if $A \subseteq \omega$ then either $A \in u$ or $\omega \setminus A \in u$. We see that u is an ultrafilter.

Now let O be any open subset of $\beta\mathbb{H}$ such that $O \cap K \neq \varnothing$; we must show that $O \setminus K$ intersects $\mathbb{H}^*$. The set $A = \{n : O \cap (a_n, b_n) \neq \varnothing\}$ is in u because O intersects K. Split A into two infinite sets A_1 and A_2. One of these sets, say A_1, is not in u. But then we may use it to find a point in $O \cap \mathbb{H}^* \setminus K$: choose $x_n \in O \cap (a_n, b_n)$ for $n \in A_1$ and consider the closure of $\{x_n : n \in A_1\}$. □

Notes for Section 4.

Theorem 4.2 was proved by WOODS in [1968] and BELLAMY in [1971]. The proof given here appears in van Douwen's notes.

5. Standard subcontinua

Looking back at the proof of Theorem 4.2 we see that we actually constructed an embedding of the space $\mathbb{M}$ into $\mathbb{H}$: the map defined by $\varphi(n,x) = a_n + x(b_n - a_n)$. Its extension $\beta\varphi : \beta\mathbb{M} \to \beta\mathbb{H}$ is also an embedding. The reader will verify without difficulty that the continuum K is contained in the continuum $\beta\varphi[\mathbb{I}_u]$.

We get one more justification for our interest in standard subcontinua: every proper subcontinuum is contained in a standard subcontinuum.

To make dealing with them a bit easier we introduce some notation. If $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ are sequences in $\mathbb{H}$ such that $a_n < b_n < a_{n+1}$ for all n and $\lim_n a_n = \infty$ then we denote the points $u{-}\lim a_n$ and $u{-}\lim b_n$ by a_u and b_u respectively; this conforms with the convention adopted in Section 2. We define a homeomorphism of $\mathbb{M}$ onto $\bigcup_n [a_n, b_n]$ as above. Furthermore, since $\mathbb{I}_u$ is irreducible between 0_u and 1_u, the continuum $\beta\varphi[\mathbb{I}_u]$ is irreducible between a_u and b_u. Because of this we shall denote $\beta\varphi[\mathbb{I}_u]$ by $[a_u, b_u]$. As we shall see, the standard subcontinua really do behave like intervals; this is another reason why we adopted the interval notation. Finally, as in the proof of Theorem 4.2, we shall write

$$F(a,b,A) = \bigcup_{n\in A} [a_n, b_n],$$

whenever $A \subseteq \omega$. We observe that

$$[a_u, b_u] = \bigcap_{U \in u} \mathrm{cl}_{\beta\mathbb{H}}\, F(a, b, U).$$

It follows that $[a_u, b_u]$ does not change if we change $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ on a set not in u. We shall use this fact quite often.

For later use we sharpen the statement that every subcontinuum is contained in a standard subcontinuum a bit.

5.1. Theorem. *If K is a proper subcontinuum of $\mathbb{H}^*$ and U a neighbourhood of K then there is a standard subcontinuum L of $\mathbb{H}^*$ such that $K \subseteq L \subseteq U$.*

The proof is implicit in the proof of Theorem 4.2; apply Proposition 3.2 to K and U and observe that the standard subcontinuum found in this way is contained in U.

The following corollary is quite handy in many situations. Using it we shall show, for example, that $\mathbb{H}^*$ is hereditarily unicoherent.

5.2. Corollary. *Let K be a proper subcontinuum of $\mathbb{H}^*$. Then K is the intersection of the family of all standard subcontinua containing it.*

The following theorem will be the key to practically the whole structure theory of the subcontinua of $\mathbb{H}^*$. Its proof is a bit technical but certainly worth the effort.

5.3. Theorem. *Let $K = [a_u, b_u]$ and $L = [c_v, d_v]$ be standard subcontinua of $\mathbb{H}^*$ with a nonempty intersection. Then one of the following three cases occurs:*

1. *$a_u, b_u \in L$; in this case $K \subseteq L$ and there is a finite-to-one map $\varphi : \omega \to \omega$ such that $\varphi(u) = v$.*
2. *$a_u \in L$ and $b_u \notin L$; then there is a permutation φ of ω such that $\varphi(u) = v$ and $K \cup L$ is the standard subcontinuum $[c_v, b_u]$. If $a_u = d_v$ then $K \cap L = \{a_u\}$, otherwise $K \cap L$ is the standard subcontinuum $[a_u, d_v]$. (The case $a_u \notin L$ and $b_u \in L$ is similar.)*
3. *$a_u, b_u \notin L$; then $c_v, d_v \in K$ and we are in Case 1 with the roles reversed.*

In short, the union of two standard subcontinua is a standard subcontinuum and so is their intersection unless it is a one-point set.

□ For every $V \in v$ we let A_V (B_V) be the set of n with $a_n \in F(c, d, V)$ $(b_n \in F(c, d, V)$ respectively). We consider three cases.

Case 1. Here we have $A_V, B_V \in u$ for all $V \in v$. We must show that $[a_u, b_u] \subseteq [c_v, d_v]$. After changing the sequences on sets not in u or v respectively we may as well assume that $a_n, b_n \in F(c, d, \omega)$ for all n.

Define $\varphi, \psi : \omega \to \omega$ by demanding that $a_n \in [c_{\varphi(n)}, d_{\varphi(n)}]$ and $b_n \in [c_{\psi(n)}, d_{\psi(n)}]$. It is clear that $\varphi(n) \leq \psi(n) \leq \varphi(n+1)$ for all n. Let $A = \{n : \varphi(n) = \psi(n)\}$.

If $A \in u$ then we get $F(a, b, A \cap A_V \cap B_V) \subseteq F(c, d, V)$ for all $V \in v$ and we conclude that $[a_u, b_u] \subseteq [c_v, d_v]$. The map φ is finite-to-one on A and it maps u to v. We can make some inessential changes to define it on the whole of ω.

The assumption $\omega \setminus A \in u$ leads to a contradiction. For let $\{i_n : n \in \omega\}$ be the monotone enumeration of $\omega \setminus A$ and put $P = \{\varphi(i_0), \psi(i_1), \varphi(i_2), \psi(i_3), \ldots\}$ and

$Q = \{\psi(i_0), \varphi(i_1), \psi(i_2), \varphi(i_3), \ldots\}$. Since the set $F\bigl(c, d, \omega \setminus (P \cup Q)\bigr)$ contains no points a_n or b_n with $n \in \omega \setminus A$ we must have $P \cup Q \in v$. On the other hand both $A_P \cap B_P$ and $A_Q \cap B_Q$ are empty, so that neither P nor Q belongs to v. This is the desired contradiction.

CASE 2. Now we have $A_V \in u$ for all $V \in v$, but some B_V is not in u.

We may assume that $a_n \in F(c, d, \omega)$ and $b_n \notin F(c, d, \omega)$ for all n. Let

$$P = \bigl\{m : \exists n \bigl[a_n \in [c_m, d_m]\bigr]\bigr\};$$

then $P \in v$ because $A_{\omega \setminus P} = \varnothing$.

The map $\varphi : \omega \to P$ defined by $a_n \in [c_{\varphi(n)}, d_{\varphi(n)}]$ is one-to-one since we must have

$$c_{\varphi(n)} \le a_n \le d_{\varphi(n)} < b_n < c_{\varphi(n+1)}$$

for all n. It follows quite easily that $v = \varphi(u)$. If we now let $e_n = c_{\varphi(n)}$ and $f_n = d_{\varphi(n)}$ for all n then we get $e_u = c_v$, $f_u = d_v$ and $[a_u, b_u] \cup [c_v, d_v] = [e_u, b_u]$.

Furthermore, if $\{n : a_n = d_{\varphi(n)}\} \in u$ then $[a_u, b_u] \cap [c_v, d_v] = \{a_u\}$ and if $\{n : a_n < d_{\varphi(n)}\} \in u$ then $[a_u, b_u] \cap [c_v, d_v] = [a_u, f_u]$.

CASE 3. Now there is $V_0 \in v$ such that $A_{V_0}, B_{V_0} \notin u$. Let $U_0 = \omega \setminus (A_{V_0} \cup B_{V_0})$. Consider any $U \in u$ contained in U_0. The set $V = \{m \in V_0 : [c_m, d_m] \cap F(a, b, U) \neq \varnothing\}$ belongs to v, but because $U \subseteq U_0$ we know that $[c_m, d_m]$ is contained in $[a_n, b_n]$ as soon as it intersects that set. We see that $F(c, d, V) \subseteq F(a, b, U)$ and we conclude that $[c_v, d_v] \subseteq [a_u, b_u]$. □

We see that the standard subcontinua really do behave like intervals. The following lemma provides further evidence. A family of sets is called *linked* if every two elements of it have a nonempty intersection. It is easy to see, for example, that every linked family of closed intervals in $\mathbb{I}$ has a nonempty intersection.

5.4. LEMMA. *The intersection of a finite linked family of standard subcontinua is a one-point set or a standard subcontinuum.*

□ We prove the lemma by induction on the number n of elements of the family.

For $n = 2$ we have Theorem 5.3; and the induction step reduces to the case $n = 3$ as follows: If $\{K_1, \ldots, K_{n+1}\}$ is linked then the family $\{K_i \cap K_{n+1} : i \le n\}$ consists of standard subcontinua and possibly a one-point set. To show that this last family is linked we apply the case $n = 3$ to every triple $\langle i, j, n+1\rangle$.

Finally then let $\{K_1, K_2, K_3\}$ be a linked family of standard subcontinua. Then $K_1 \cup K_2$ is a standard subcontinuum and hence so is $(K_1 \cup K_2) \cap K_3$. But then the closed sets $K_1 \cap K_3$ and $K_2 \cap K_3$ must intersect and since these are standard subcontinua the intersection $K_1 \cap K_2 \cap K_3$ is a standard subcontinuum or a one-point set. □

5.5. COROLLARY. *The intersection of every linked family of standard subcontinua is a continuum.*

□ By the lemma every finite intersection from the family is again a standard subcontinuum, or a singleton. So the intersection is the intersection of a downward directed family of continua, hence a continuum. □

This corollary allows us to conclude that $\mathbb{H}^*$ is hereditarily unicoherent. We recall that a continuum X is said to be *unicoherent* if whenever it is written as the union of two subcontinua — $X = K \cup L$ — the intersection of K and L is connected. A *hereditarily unicoherent continuum* is one in which every subcontinuum is unicoherent, equivalently in which the intersection of any two subcontinua is connected if nonempty. Using the same argument as in the proof of Corollary 5.5 we can prove that in a hereditarily unicoherent continuum the intersection of every linked family of subcontinua is again a continuum.

5.6. THEOREM. *The continuum $\mathbb{H}^*$ is hereditarily unicoherent.*

□ Let K and L be subcontinua of $\mathbb{H}^*$ with nonempty intersection. Since $\mathbb{H}^*$ is indecomposable we may assume that K and L are both proper subcontinua. By Corollary 5.2 we may take families $\mathcal{L}_K$ and $\mathcal{L}_L$ of standard subcontinua with intersection K and L respectively. Then $\mathcal{L}_K \cup \mathcal{L}_L$ is a linked family of standard subcontinua with intersection $K \cap L$ so that $K \cap L$ is a continuum. □

We now prove a nice structure theorem for decomposable subcontinua of $\mathbb{H}^*$. To begin we note that standard subcontinua and their non-degenerate intervals are decomposable (as we will see later layers are indecomposable). The structure theorem shows that these are the only decomposable subcontinua of $\mathbb{H}^*$. We begin with a lemma.

5.7. LEMMA. *Let K and L be proper subcontinua of $\mathbb{H}^*$ and assume that the sets $K \cap L$, $K \setminus L$ and $L \setminus K$ are nonempty. Then there is a standard subcontinuum of $\mathbb{H}^*$ such that K and L are non-degenerate intervals of it.*

□ Fix points $x \in K \setminus L$ and $y \in L \setminus K$. Take standard subcontinua K^+ and L^+ around K and L respectively such that $x \notin L^+$ and $y \notin K^+$. By Theorem 5.3 the union $K^+ \cup L^+$ is a standard subcontinuum, denote it by $[a_u, b_u]$. Now x is not in L^+, but L^+ contains $K \cap L$. It follows that K meets two different layers of $[a_u, b_u]$: the layer of x and a layer in L^+. Now apply Theorem 2.11 to K. The argument for L is the same of course. □

5.8. THEOREM. *Let K be a subcontinuum of $\mathbb{H}^*$. Then either K is indecomposable or there is a standard subcontinuum such that K is a non-degenerate interval of it. In particular if K is decomposable then K is irreducible between two points and it has a dense set of cut points.*

□ If K is decomposable then the previous lemma immediately implies that K is a non-degenerate interval of some standard subcontinuum. □

The following consequence of Lemma 5.7 will play a role in our investigation of cut points of subcontinua of $\mathbb{H}^*$.

5.9. THEOREM. *If K and L are subcontinua of $\mathbb{H}^*$ that intersect and if one of K and L is indecomposable then $K \subseteq L$ or $L \subseteq K$.*

An easy consequence of this theorem is the following.

5.10. THEOREM. *Let K and L be subcontinua of $\mathbb{H}^*$ such that K is a proper subset of L and L is indecomposable. Then there is a standard subcontinuum M such that $K \subseteq M \subseteq L$.*

$\square$ Take $x \in L \setminus K$ and let M be a standard subcontinuum around K such that $x \notin M$. Since clearly M intersects L and L is not contained in M we must have $M \subseteq L$. $\square$

5.11. COROLLARY. *Every subcontinuum of $\mathbb{H}^*$ contains a standard subcontinuum and hence no subcontinuum of $\mathbb{H}^*$ is hereditarily indecomposable.*

$\square$ The first part follows from Theorems 5.8 and 5.10. The 'hence' is justified by the fact that standard subcontinua are decomposable. $\square$

Notes for Section 5.

The results in this section are all taken from van Douwen's notes. Standard subcontinua appear implicitly in GILLMAN and JERISON [1976, 10N].

Theorem 5.6 was proved by GILLMAN and HENRIKSEN in [1956, Corollary 4.10] by algebraic means; they established an algebraic property of the ring $C^*(\mathbb{R})$ and showed that, in general, a normal space X is hereditarily unicoherent if $C^*(X)$ has this property.

Some of the results of this section have been rediscovered in recent years.

The material in this section also appears in ZHU [19∞a], with different proofs.

The fact that no subcontinuum of $\mathbb{H}^*$ is hereditarily indecomposable was established by SMITH in [1987a] using a somewhat more complicated argument than the one presented here. In [1988] SMITH showed that much more is true: no power of $\mathbb{H}^*$ contains a hereditarily indecomposable subcontinuum. This is in strong contrast with the metric situation: it was proved by BING in [1951] that every two-dimensional metric continuum contains a hereditarily indecomposable subcontinuum. It is also in contrast with the situation for $(\mathbb{R}^2)^*$: It was shown by SMITH in [1987b] that if $X = \bigoplus_{n\in\omega} K_n$ is a sum of hereditarily indecomposable continua then every continuum K_u is hereditarily indecomposable as well. Apply this to a discrete (infinite) collection of pseudoarcs in the plane; one gets hereditarily indecomposable subcontinua in $(\mathbb{R}^2)^*$.

6. Layers and other indecomposable continua

This section is devoted to the study of some properties of layers in standard subcontinua. As a byproduct of these investigations we get insight in the structure of the indecomposable subcontinua of $\mathbb{H}^*$ as well.

Let L be a layer of $\mathbb{I}_u$. We let A_L be the set of sequences $\langle a_n\rangle_n$ in $\mathbb{I}$ for which $a_u <_u L$; if $L = \{x_u\}$ for some $x_u \in P_u$ then $A_L = \{\langle a_n\rangle_n : a_u < x_u\}$, otherwise $A_L = \{\langle a_n\rangle_n : L \subseteq [a_u, 1_u]\}$. Likewise B_L denotes the set of sequences $\langle b_n\rangle_n$ for which $L <_u b_u$.

The continuity properties of $<_u$ imply that

$$L = \bigcap\{[a_u, b_u] : \langle a_n\rangle_n \in A_L \text{ and } \langle b_n\rangle_n \in B_L\}.$$

From this it follows that if L is not of the form $\{x\}$ with $x \in P_u$ then the pair $\langle A_L, B_L\rangle$ determines a gap in P_u, i.e., there is no sequence $\langle c_n\rangle_n$ such that $a_u <_u c_u <_u b_u$ for all $\langle a_n\rangle_n \in A_L$ and $\langle b_n\rangle_n \in B_L$.

We can use A_L and B_L to identify a nice local base at L in $\beta\mathbb{M}$.

6.1. Lemma. *If O is open in $\beta\mathbb{M}$ and $L \subseteq O$ then there are $\langle a_n \rangle_n \in A_L$, $\langle b_n \rangle_n \in B_L$ and $U \in u$ such that*

$$L \subseteq \mathrm{cl}_{\beta\mathbb{M}} F(a, b, U) \subseteq O.$$

□ This follows by compactness; L is the intersection of sets of the desired form. □

From now on we let L be a layer not of the form $\{x\}$ for any $x \in P_u$. We shall see that L is an indecomposable continuum; this is trivial if L is a one-point set; to show it for the other layers will require some work.

We begin by analyzing the situation in Case 1 of Theorem 5.3 more carefully. We have two standard subcontinua $[a_u, b_u]$ and $[c_v, d_v]$ such that $[a_u, b_u] \subseteq [c_v, d_v]$. We also have a finite-to-one map $\varphi : \omega \to \omega$ such that $\varphi(u) = v$. The proof of Theorem 5.3 shows that, without loss of generality, $[a_n, b_n] \subseteq [c_{\varphi(n)}, d_{\varphi(n)}]$ for all n. Next we apply Theorem 5.8 to deduce that $[a_u, b_u]$ is either contained in a layer of $[c_v, d_v]$ or that it is an interval of $[c_v, d_v]$. The following lemma tell us how we can see which case holds by looking at the map φ.

6.2. Lemma. *The map φ is one-to-one on some element of u if and only if $[a_u, b_u]$ is an interval of $[c_v, d_v]$.*

□ Suppose φ is one-to-one on some element U of u, without loss of generality $U = \omega$. Consider the sequences $\langle e_n \rangle_n$ and $\langle f_n \rangle_n$ defined by $e_n = c_{\varphi(n)}$ and $f_n = d_{\varphi(n)}$.

It should be clear that $e_u = c_v$, $f_u = d_v$ and $[e_u, f_u] = [c_v, d_v]$. After this reindexing it follows immediately that $[a_u, b_u]$ is an interval of $[c_v, d_v]$.

Now assume that φ is one-to-one on *no* element of u. Let $\langle x_n \rangle_n$ be any sequence with $x_n \in [c_n, d_n]$ for all n. Divide ω into three sets as follows:

$$\begin{aligned}
&U_1 = \{i : \text{if } \varphi(i) = n \text{ then } x_n < a_i\},\\
&U_2 = \{i : \text{if } \varphi(i) = n \text{ then } x_n > b_i\}, \text{ and}\\
&U_3 = \omega \setminus (U_1 \cup U_2).
\end{aligned}$$

Observe that φ is one-to-one on U_3 (if $i \in U_3$ then $x_{\varphi(i)} \in [a_i, b_i]$), so that either $U_1 \in u$ or $U_2 \in u$. But this means that either $[a_u, b_u] \subseteq [c_v, x_v]$ or $[a_u, b_u] \subseteq [x_v, d_v]$.

Since this holds for all sequences $\langle x_n \rangle_n$ we conclude that $[a_u, b_u]$ is contained in a layer of $[c_v, d_v]$. □

Now we are ready to show that non-trivial layers of standard subcontinua are indecomposable.

6.3. Theorem. *Every layer of $\mathbb{I}_u$ is an indecomposable continuum.*

□ Since one-point layers are trivially indecomposable we consider a layer L with at least two points. A consequence of Theorem 5.8 is that a decomposable continuum contains a standard subcontinuum with (relative) nonempty interior. We see that it suffices to show that every standard subcontinuum of L is nowhere dense in L.

Let $\langle [c_i, d_i] \rangle_i$ be a sequence of intervals in $\mathbb{M}$ and let $v \in \omega^*$ be such that $[c_v, d_v] \subseteq L$. Let φ be the map determined by $[c_i, d_i] \subseteq \mathbb{I}_{\varphi(i)}$. By Lemma 6.2 the map φ is not one-to-one on any element of v.

To begin we observe that for every $\langle a_n\rangle_n \in A_L$, every $\langle b_n\rangle_n \in B_L$ and every $U \in u$ the set

$$\{i : [c_i, d_i] \subseteq F(a, b, U)\}$$

is in v. Next we consider for $V \in v$ and $\langle a_n\rangle_n$, $\langle b_n\rangle_n$ and U as above the sequence of numbers $\langle m_n\rangle_{n\in U}$ defined by

$$m_n = \left|\{i \in V : [c_i, d_i] \subseteq [a_n, b_n]\}\right|.$$

This sequence must be unbounded since otherwise we could divide V into finitely many sets on which φ would be one-to-one; but none of these sets would be in v.

Now let O be open in $\mathbb{M}$ such that $O \cap [c_v, d_v] \neq \varnothing$. The set $V_O = \{i : O \cap [c_i, d_i] \neq \varnothing\}$ belongs to v. We split V_O into two infinite pieces V_0 and V_1 as follows. In every interval $\mathbb{I}_n$ that contains some of the $[c_i, d_i]$ these intervals lie ordered by the order of $\mathbb{I}_u$ (formally, if $[c_i, d_i], [c_j, d_j] \subseteq \mathbb{I}_n$ say $i <_n j$ iff $\max\{c_i, d_i\} < \min\{c_j, d_j\}$). For such n put into V_0 those i which are even numbered in $<_n$ and put the odd numbered i into V_1. Using the above observation on unbounded sequences it is now straightforward to show that the closures of $O \cap \bigcup_{i\in V_0}[c_i, d_i]$ and $O \cap \bigcup_{i\in V_1}[c_i, d_i]$ both meet L. One of these sets is disjoint from $[c_v, d_v]$.

Since L is a regular topological space this finishes the proof. □

We can also show that there are indecomposable subcontinua of $\mathbb{H}^*$ that are not layers of standard subcontinua. The construction relies on the following lemma.

6.4. LEMMA. *Let $\mathcal{K}$ be a linearly ordered family of indecomposable subcontinua of $\mathbb{H}^*$, and let $L = \bigcap \mathcal{K}$. Then L is indecomposable and if L is a layer of some standard subcontinuum then $L \in \mathcal{K}$.*

□ Let M be any standard subcontinuum containing L. By Theorems 5.8 and 5.9 every $K \in \mathcal{K}$ either contains M or is contained in a layer of M.

This immediately implies that L is contained in a layer of every standard subcontinuum containing it and hence indecomposable. It also implies that $L \in \mathcal{K}$ if L is a layer of some M, because some $K \in \mathcal{K}$ has to be contained in M hence in a layer, but this layer must be L. □

6.5. THEOREM. *There is an indecomposable subcontinuum of $\mathbb{H}^*$ that is not a layer of any standard subcontinuum.*

□ Apply Proposition 2.12 and Theorem 5.10 infinitely many times to find a sequence $\langle K_n\rangle_n$ of continua such that K_{n+1} is a non-trivial layer of a standard subcontinuum of K_n. By Theorem 6.3 every K_n is indecomposable.

By the previous lemma the intersection $\bigcap_n K_n$ is an indecomposable continuum that is not a layer of any standard subcontinuum. □

Note that $\bigcap_n K_n$ can not be a one-point set either: pick $x_n \in K_n \setminus K_{n+1}$ for every n and let $D = \{x_n : n \in \omega\}$. The set D is relatively discrete so its closure is homeomorphic with $\beta\omega$, but also $\overline{D} \setminus D \subseteq \bigcap_n K_n$.

Notes for Section 6.

Lemma 6.1 was proved by MIODUSZEWSKI in [1978]; using it Mioduszewski showed that the decomposition of $\mathbb{M}^*$ into layers is upper-semicontinuous (cf. Corollary 2.10).

Theorem 6.3 appears in van Douwen's notes, but without proof. van Douwen used it to deduce Theorem 6.5.

Theorem 6.3 was rediscovered by SMITH in [19∞] and ZHU in [1991b]. In [19∞a] ZHU rediscovered Theorem 6.5.

7. Cut points and nonhomogeneity of $\mathbb{H}^*$

In this section we will give a clear cut reason why $\mathbb{H}^*$ is not homogeneous, by exhibiting two points with easily defined different topological behaviour.

A point of a continuum X is said to be a *weak cut point* of X if it is a cut point of some subcontinuum of X.

Our goal then will be to find a weak cut point of $\mathbb{H}^*$ and another point that is not a weak cut point.

A large family of weak cut points is the following. We shall call a point x of $\mathbb{H}^*$ a *near point* if it is in the closure of some closed and discrete subset of $\mathbb{H}$. If a point is not near then we call it a *far point*.

The following theorem is a direct consequence of Theorem 2.5.

7.1. THEOREM. *Every near point of $\mathbb{H}$ is a weak cut point of $\mathbb{H}^*$.*

It remains to find a point that is not a weak cut point. The following lemma shows that we only have to worry about standard subcontinua.

7.2. LEMMA. *Let z be a weak cut point of $\mathbb{H}^*$. Then z is a cut point of some standard subcontinuum.*

□ Let K be a subcontinuum such that z is a cut point of K. Then K is decomposable and hence a non-degenerate interval in some standard subcontinuum M. But then z is also a cut point of M. □

The following proposition tells us how we can recognize cut points of standard subcontinua by looking at their layers.

7.3. PROPOSITION. *A point x of $\mathbb{I}_u \setminus \{0_u, 1_u\}$ is a cut point if and only if its layer is a one-point set.*

□ If $L_x = \{x\}$ then $\bigl\{[0_u, L_x), (L_x, 1_u]\bigr\}$ is a partition of $\mathbb{I}_u \setminus \{x\}$ into disjoint open sets.

On the other hand if $L_x \neq \{x\}$ then $\overline{[0_u, L_x)} \setminus \{x\}$ and $\overline{(L_x, 1_u]} \setminus \{x\}$ have a nonempty intersection. Since both sets are connected their union $\mathbb{I}_u \setminus \{x\}$ is connected as well. □

We see that a point is a weak cut point iff it determines a one-point layer in some standard subcontinuum.

To find a point that is not a weak cut point we consider a maximal chain $\mathcal{K}$ of non-degenerate indecomposable subcontinua of $\mathbb{H}^*$. The intersection of $\mathcal{K}$ consists of exactly one point. To see this we note that $\bigcap\mathcal{K}$ is indecomposable by Lemma 6.4. If it were non-degenerate then we could find a standard subcontinuum of it and add a non-trivial layer of this continuum to the family $\mathcal{K}$, contradicting the maximality. Let y be the point of $\bigcap\mathcal{K}$.

7.4. THEOREM. *The point y is not a weak cut point.*

$\square$ We must show that whenever M is a standard subcontinuum of $\mathbb{H}^*$ that contains y, the layer of y in M is non-degenerate.

So let M be standard with $y \in M$. Using Theorem 5.9 we deduce that there must be a $K \in \mathcal{K}$ such that $K \subseteq M$. But this K must be contained in a layer of M, and it must be the layer of y which therefore is non-trivial. $\square$

The converse of this theorem also holds: if y is not a weak cut point then it is the intersection of a chain of nondegenerate indecomposable continua. By Theorem 5.9 the family $\mathcal{K}_y$ of nondegenerate indecomposable continua that contain y is a chain. By Lemma 6.4 the intersection of $\mathcal{K}_y$ is indecomposable. To see that the intersection is exactly $\{y\}$ we combine Theorem 5.10, Lemma 7.2, Proposition 7.3 and Theorem 6.3.

It is interesting to note that non weak cut points can also be constructed directly from $\mathbb{H}$. For this we consider the family

$$\mathcal{F} = \{F : F \text{ is closed and } m(\mathbb{H} \setminus F) < \infty\}.$$

Thus $\mathcal{F}$ consists of those closed sets whose complement has finite Lebesgue measure. It is clear that $\mathcal{F}$ is a filter of closed sets. A point of $\mathbb{H}^*$ that extends this filter is called a *large point*. Since closed and discrete subsets of $\mathbb{H}$ are countable it follows right away that large points are far.

We shall show that large points are not weak cut points either. In the proof we use the following characterization, in terms of A_L and B_L, of when L is a one-point set. This characterization is somewhat more amenable to set theory than the result from Proposition 7.3. We shall use it in Section 8.

We recall that a *null sequence* is a sequence that converges to zero.

7.5. PROPOSITION. *The layer L consists of one point if and only if for every positive null sequence $\langle x_n \rangle_n$ in $(0,1)$ there are $\langle a_n \rangle_n \in A_L$ and $\langle b_n \rangle_n \in B_L$ such that $\{n : b_n - a_n < x_n\} \in u$.*

$\square$ With every null sequence $\langle x_n \rangle_n$ we associate three closed sets F_0, F_1 and F_2 that cover $\mathbb{M}$, as follows. Choose a sequence $\langle k_n \rangle_n$ in $\mathbb{N}$ such that $1/k_n < x_n$. For $n \in \omega$ and $i < 3$ we let

$$F_{i,n} = \bigcup \Big\{ \big[(3j+i)/3k_n, (3j+i+1)/3k_n\big] : j < k_n \Big\}.$$

The sets $F_{i,n}$ divide the interval $\mathbb{I}$ into three pieces. We let $F_i = \bigcup_n F_{i,n}$ for $i < 3$.

If $L = \{p\}$ is a one-point layer and $\langle x_n \rangle_n$ is a null sequence then one of the sets F_i, say F_0, is not in p. We can therefore find a set of the form $F(a,b,U)$, with $\langle a_n \rangle_n \in A_L$ and $\langle b_n \rangle_n \in B_L$, that is disjoint from F_0. But this readily implies that $b_n - a_n < 2x_n/3$ for all $n \in U$.

If L contains two distinct points p and q take elements F and G of p and q respectively that are disjoint. The set $U = \{n : F \cap \mathbb{I}_n \neq \varnothing \neq G \cap \mathbb{I}_u\}$ belongs to u. Now if $\langle a_n \rangle_n \in A_L$ and $\langle b_n \rangle_n \in B_L$ then the set V of those n for which $[a_n, b_n]$ meets both $F \cap \mathbb{I}_n$ and $G \cap \mathbb{I}_u$ also belongs to u. However for these n we must have $b_n - a_n \geq d(F \cap \mathbb{I}_n, G \cap \mathbb{I}_n)$; we see that any null sequence $\langle x_n \rangle_n$ satisfying $x_n < d(F \cap \mathbb{I}_n, G \cap \mathbb{I}_n)$ will do. $\square$

7.6. Theorem. *A large point is not a cut point of any standard subcontinuum.*

□ Let z be a large point and let M be a standard subcontinuum containing it. Let L be the layer of z in M. Since every element of z must have infinite measure we can apply Proposition 7.5 with any convergent series to conclude that L is not a one-point set. □

Notes for Section 7.

Theorems 7.1 and 7.4 are taken from van Douwen's notes. In [19∞a] Zhu rediscovered Theorem 7.4 and showed that its converse holds.

Proposition 7.5 is a variation of Lemma 2.1 of Zhu [19∞b].

In [1980] van Mill and Mills showed that near points have a slightly stronger cut point property than the one given here: there are a subcontinuum K and a neighbourhood U of K such that the point is a cut point of every continuum between K and U. They then went on to show that large points do not have this property. Note however that, by Lemma 7.2, this property is only formally stronger.

Large points were given as examples of far points by Fine and Gillman in [1962]; they credited W. F. Eberlein with the construction. As a point of interest we note that large points are not remote: consider the complement of a dense open set of measure 1.

8. The existence of non-trivial cut points

The reader may certainly have wondered why Corollary 7.3 does not say that a point is a cut point of $\mathbb{I}_u$ iff it is an element of P_u, and likewise why Lemma 7.2 does not simply identify the weak cut points as the near points of $\mathbb{H}$. The answer is: 'because that statement is not true in general'. As we shall see in Theorem 8.3, Martin's Axiom for Countable posets (MAC) implies the existence of a point u of ω^* for which there are cut points of $\mathbb{I}_u$ outside P_u. When mapped into $\mathbb{H}^*$ via some embedding, these become weak cut points that are not near.

Let us call cut points of $\mathbb{I}_u$ that are not points of P_u *non-trivial cut points* of $\mathbb{I}_u$.

In our investigations into the nature of cut points we shall need the concept of a remote point. A point of X^* is said to be a *remote point* of X if it is not in the closure—in βX—of any nowhere dense subset of X. It is well-known that $\mathbb{R}$ has remote points. We shall give a proof of this fact later in this section for completeness and to contrast it with the proof of Theorem 8.3.

Let us investigate what non-trivial cut points should look like. By Lemma 6.1 we know convenient local bases for non-trivial cut points. Let us restate this lemma in terms of closed sets: If x is a non-trivial cut point and F is a closed set not in x then there are $\langle a_n \rangle_n \in A_{\{x\}}$, $\langle b_n \rangle_n \in B_{\{x\}}$ and $U \in u$ such that $F(a, b, U)$ is disjoint from F. The converse is also true: if the sets $F(a, b, U)$ determine a local base at x then x is a cut point because these sets are a local base at the layer of x.

This does not quite mean that a cut point, when viewed as a closed ultrafilter, must have a base of sets of the form $F(a, b, U)$, as we shall see momentarily. Let us note that a non-trivial cut point that has a base of sets of the form $F(a, b, U)$ must be a remote point. The converse is also true as the following result shows.

8.1. Proposition. *A non-trivial cut point is a far point of* $\mathbb{M}$. *It is a remote point if and only if it has a base consisting of sets of the form* $F(a, b, U)$.

$\square$ Let x be a non-trivial cut point of $\mathbb{I}_u$ and let $D \subseteq \mathbb{M}$ be closed and discrete. For every n the set $D \cap \mathbb{I}_n$ is finite, enumerate it as $\{d_{n,i} : i < k_n\}$ (in increasing order). Put $D_e = \{d_{n,2i} : i \leq k_n/2\}$ and $D_o = D \setminus D_e$. One of these sets, say D_o, is not in x. There is a neighbourhood $F(a,b,U)$ of x disjoint from D_o. Because each interval (a_n, b_n) is convex it contains at most one point of D; but x is not in P_u so now we can find a neighbourhood of it that is disjoint from D.

To prove the second statement assume x is remote and let $F \in x$. We must find U, $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ such that $F(a,b,U) \subseteq F$. Since x is remote it is not in the closure of the boundary of F. Take a neighbourhood $F(a,b,U)$ of x that is disjoint from the boundary of F. We can assume that $[a_n, b_n]$ meets F for every $n \in U$. But since $[a_n, b_n]$ is connected and meets F but not its boundary it can not meet the complement of F as well. We conclude that $F(a,b,U) \subseteq F$. $\square$

Let us call a non-trivial cut point with a base of sets of the form $F(a,b,U)$ a *narrow remote point*. This name is inspired by Proposition 7.5, and also by the proof of Theorem 8.4 because the remote points constructed there will be anything but narrow: their elements tend to be spread out more and more over the intervals $\mathbb{I}_n$.

We shall now construct two kinds of non-trivial cut points in some $\mathbb{I}_u$, one will be remote the other will not. We construct the points from a special remote point of the space $\omega \times \mathbb{C}$; this remote point will have a base of clopen sets.

Let $\mathcal{B}$ be the canonical clopen base for $\mathbb{C}$. This base is indexed by the set $\mathbb{S} = \bigcup_{n\in\omega} {}^n2$ of all finite 0-1-valued sequences: if $s \in \mathbb{S}$ then $B_s = \{x \in \mathbb{C} : s \subset x\}$.

8.2. Theorem. MAC *implies that there are a free ultrafilter u on ω, and a point x in $(\omega \times \mathbb{C})^*$ such that x has a base consisting of sets of the form*

$$S(U,f) = \bigcup_{n\in U} \{n\} \times B_{f(n)},$$

where $U \in u$ and f is a function from ω to $\mathbb{S}$.

$\square$ Let $\langle F_\alpha : \alpha < \mathfrak{c} \rangle$ count the family of closed subsets of $\omega \times \mathbb{C}$. By induction on α we shall find U_α and f_α such that the family $\{S(U_\alpha, f_\alpha) : \alpha < \mathfrak{c}\}$ has the finite intersection property and such that for every α

$$S(U_\alpha, f_\alpha) \subseteq F_\alpha \text{ or } S(U_\alpha, f_\alpha) \cap F_\alpha = \varnothing. \qquad (*)$$

To avoid having to check irrelevant details we assume that $F_n = \omega \times \mathbb{C}$, $U_n = [n, \omega)$ and $f_n(i) = \langle\rangle$ for all $n < \omega$ (and all i). We start the induction at ω. Let $\gamma \geq \omega$ and assume everything has been found up to but not including γ. We are to construct U_γ and f_γ. We shall use the subposet $\mathbb{P}_\gamma$ of $\mathrm{Fn}(\omega, \mathcal{B})$ consisting of those finite partial functions p from ω into $\mathcal{B}$ that satisfy

$$\forall n \in \mathrm{dom}(p)\ [B_{p(m)} \subseteq F_\gamma \text{ or } B_{p(m)} \cap F_\gamma = \varnothing]. \qquad (**)$$

This poset is clearly countable. We define some dense subsets of $\mathbb{P}_\gamma$: for a finite subset E of γ and an $n \in \omega$ we let $D_{E,n}$ be the set of those elements of $\mathbb{P}$ for which there is an $m \in \mathrm{dom}(p)$ such that $m \geq n$, $m \in \bigcap_{\alpha\in E} U_\alpha$ and

$$B_{p(m)} \subseteq \bigcap_{\alpha\in E} B_{f_\alpha(m)}.$$

The set $D_{E,n}$ is easily seen to be dense: there are infinitely many m for which the intersection

$$\bigcap_{\alpha\in E} S(U_\alpha, f_\alpha) \cap \bigl(\{m\} \times \mathbb{C}\bigr)$$

is nonempty. Now let G be a filter on $\mathbb{P}_\gamma$ that meets all sets of the form $D_{E,n}$ and let $f_\gamma = \bigcup G$. Clearly f_γ is a function that satisfies $(**)$. Let $A_\gamma = \{m \in \mathrm{dom}(f_\gamma) : B_{f_\gamma(m)} \subseteq F_\gamma\}$ and $B_\gamma = \mathrm{dom}(f_\gamma) \setminus A_\gamma$. We claim that one of A_γ and B_γ can be used as U_γ. Clearly both $S(A_\gamma, f_\gamma)$ and $S(B_\gamma, f_\gamma)$ satisfy $(*)$. By construction the family

$$\bigl\{S(U_\alpha, f_\alpha) : \alpha < \gamma\bigr\} \cup \bigl\{S(\mathrm{dom}(f_\gamma), f_\gamma)\bigr\}$$

has the finite intersection property; but then so does at least one of the families that we get when we replace $\mathrm{dom}(f_\gamma)$ by A_γ or B_γ respectively. This concludes the induction step and the construction of u and x. □

We now use u and x to create non-trivial cut points that are externally different: one is a remote point of $\mathbb{M}$, the other is not.

8.3. Theorem. *If u and x are as in the previous theorem then there are non-trivial cut points y and z of $\mathbb{I}_u$ such that y is a remote point of $\mathbb{M}$ and z is not remote.*

□ Consider the following two maps from $\mathbb{C}$ to $\mathbb{I}$: $o(x) = \sum_{n\in\omega} x_n \cdot 2^{-(n+1)}$ and $e(x) = \sum_{n\in\omega} 2x_n \cdot 3^{-(n+1)}$. Of course o is the usual two-to-one surjection of $\mathbb{C}$ onto $\mathbb{I}$ and e is the usual homeomorphism of $\mathbb{C}$ onto the familiar middle-third set. We shall simply identify $\mathbb{C}$ and the middle-third set and pretend that e is the identity map. The crucial property of both o and e is that for every $s \in \mathbb{S}$ the sets $o[B_s]$ and $e[B_s]$ are order-convex in $\mathbb{I}$ and $\mathbb{C}$ respectively.

Consider the maps from $\omega \times \mathbb{C}$ to $\mathbb{M}$ induced by o and e; we shall denote these by o and e also. We claim that the points $y = \beta e(x)$ and $z = \beta o(x)$ are as required.

We deal with y first. Let $F \subseteq \mathbb{M}$ be closed and assume that $F \notin y$. We may choose $U \in u$ and a function f such that $F \cap S(U, f) = \varnothing$. Take $n \in U$ and let $C_{f(n)}$ be the convex hull of $B_{f(n)}$ in $\mathbb{I}_n$ and note that $B_{f(n)} = C_{f(n)} \cap \mathbb{C}$. We claim that $F \cap C_{f(n)}$ is covered by finitely many complementary intervals of $\mathbb{C}_n$. This follows from the fact that the intersection $F \cap C_{f(n)} \cap \mathbb{C}_n = F \cap B_{f(n)}$ is empty. The finite set of endpoints of these intervals we denote by w_n. The set $W = \bigcup_{n\in U} w_n$ is closed and discrete in $\omega \times \mathbb{C}$ and hence there are $V \in u$ and g such that $S(V, g) \subseteq S(U, f) \setminus W$. Take the convex hull of $S(V, g)$, this set is of the form $S(V, a, b)$, disjoint from F and an element of y.

We conclude that y is a non-trivial cut point of $\mathbb{I}_u$. Since y is in the closure of $\omega \times \mathbb{C}$, it is not a remote point of $\mathbb{M}$.

In the same way it can be shown that z has a base consisting of sets of the form $F(a, b, U)$ (the images of the sets $S(U_\alpha, f_\alpha)$), so that it is a non-trivial cut point of $\mathbb{I}_u$ that is also a remote point of $\mathbb{M}$. The only fact that really needs checking is that $\beta o^{\leftarrow}(z) = \{x\}$.

Let $t \neq x$, and let $F \in t$ and $S(U, f) \in x$ be disjoint. Take an element $S(V, g)$ of x, contained in $S(U, f)$ that contains none of the endpoints of $S(U, f)$ (the endpoints in the order-theoretic sense). A moment's reflection should convince us that $F \cap o^{\leftarrow}\big[o[S(V, g)]\big] = \varnothing$, and this implies that $\beta o(t) \neq x$. $\square$

Now that we know that non-trivial cut points may exist the question arises whether they really exist. The full answer to this question is not known yet. What we do know is that narrow remote points need not exist: there are none in Laver's model for the Borel Conjecture.

For those who are familiar with Laver forcing the following should suffice as a sketch of the argument.

Let $\langle \mathbb{P}_\alpha : \alpha < \omega_2 \rangle$ be a countable support iteration of Laver forcing. Assume that in M_{ω_2} there is a narrow remote point x, say $x \in \mathbb{I}_u$. There is an $\alpha < \omega_2$ such that, in M_α, $x \restriction \alpha$ is a narrow remote point. Let $\langle A, B \rangle$ be the gap in $P_{u \restriction \alpha}$ that determines the layer $\{x \restriction \alpha\}$ in $\mathbb{I}_{u \restriction \alpha}$.

Consider any sequence $\langle c_n \rangle_n$ in $\mathbb{I}$, from M_{ω_2}. There is a closed nowhere dense set C of $\mathbb{M}$, coded in M_α, such that $\langle n, c_n \rangle \in C$ for all n. Because $x \restriction \alpha$ is narrow we can find $a \in A$, $b \in B$ and $U \in u \restriction \alpha$ such that $F(a, b, U) \cap C = \varnothing$. We conclude that $\langle A, B \rangle$ will determine, in M_{ω_2}, a gap in P_u; hence it must determine the layer $\{x\}$.

On the other hand, if $f : \omega \to \omega$ is the Laver real added at stage α then for every null sequence x from M_α there is N such that $x_n > 1/f(n)$ for all $n > N$. It now follows from Proposition 7.5 that $\langle A, B \rangle$ can not determine a one-point layer in M_{ω_2}.

This shows that something like MAC is needed in the proof of Theorem 8.2.

To finish this section we will give a proof that $\mathbb{R}$ has remote points. The reader should contrast these 'real' remote points with narrow remote points.

8.4. THEOREM. *The space $\mathbb{R}$ has remote points.*

$\square$ Let $\mathcal{D}$ denote the family of all closed nowhere dense subsets of $\mathbb{R}$. Our aim is to find a family $\mathcal{F} = \{F_D : D \in \mathcal{D}\}$ of closed sets with the finite intersection property and such that $F_D \cap D = \varnothing$ for all $D \in \mathcal{D}$. Any closed ultrafilter extending $\mathcal{F}$ must be a remote point. We shall construct the sets F_D in pieces, as follows.

To begin we let $\mathcal{B} = \{B_i : i \in \omega\}$ be the set of open intervals with rational intervals. Furthermore we take a discrete family $\{I_n : n \in \omega\}$ of open intervals in $\mathbb{R}$, for example $I_n = (2n, 2n+1)$. For every $D \in \mathcal{D}$ and every $n \in \omega$ we shall find a closed set $F_{D,n} \subseteq O_n$ such that $F_{D,n} \cap D = \varnothing$.

After this we shall let $F_D = \bigcup_{n \in \omega} F_{D,n}$. To ensure that the resulting family $\mathcal{F}$ has the finite intersection property we shall—for a fixed n—construct the $F_{D,n}$ in such a way that every n of them have a nonempty intersection (in other words $\{F_{D,n} : D \in \mathcal{D}\}$ has the *n-intersection property*). That this suffices is readily seen.

It remains to construct the sets $F_{D,n}$. These will be made up of closures of elements of $\mathcal{B}$. We collect the possible candidates:

$$K(D, n) = \{i \in \omega : \overline{B}_i \cap D = \varnothing \text{ and } \overline{B}_i \subseteq I_n\}.$$

We shall choose, for each D, a natural number $i_{D,n}$ and we shall put

$$F_{D,n} = \bigcup\{\overline{B}_i : i \in K(D,n) \text{ and } i \le i_{D,n}\}.$$

We do this in an $n+1$-step 'closing-off argument': For a fixed D let $i_{D,0} = \min K(D,n)$ and given $i_{D,m}$ for $m < n$ determine $i_{D,m+1}$ as follows: For every $s \in \omega$ with $\overline{B}_s \subseteq O_n$ there is a $t_s \in K(D,n)$ with $\overline{B}_{t_s} \subseteq B_s$ because D is nowhere dense. Let $i_{D,m+1}$ be the first element of $K(D,n)$ above $i_{D,m}$ and all t_s for $s \le i_{D,m}$.

We verify the n-intersection property. Let $\{D_i : i < n\}$ be a family of closed nowhere dense subsets of X. We assume that the indexing is such that $i_{D_j,j} \le i_{D_k,j}$ whenever $j < k$. Now we set $s(0) = i_{D_0,0}$, and given $s(j) \le i_{D_j,j}$ we let $s(j+1) = \min\{t \in K(D_{j+1},n) : \overline{B}_t \subseteq B_{s(j)}\}$. Since $s(j) \le i_{D_j,j} \le i_{D_{j+1},j}$ we see that $s(j+1) \le i_{D_{j+1},j+1}$. We get a decreasing sequence

$$B_{s(0)} \supseteq B_{s(1)} \supseteq \cdots \supseteq B_{s(n-1)},$$

and we conclude that

$$B_{s(n-1)} \subseteq \bigcap_{j<n} F_{D_j,n}.$$

Of course $B_{s(n-1)} \ne \varnothing$. □

In the next section we will apply the knowledge acquired in this section to solve a long outstanding problem from the theory of remote points: we shall construct an autohomeomorphism of $\mathbb{H}^*$ that maps a remote point to a near point.

Notes for Section 8.

Proposition 8.1 was proved by Zhu in [1991b].

The proof of Theorem 8.2 is a minor modification of the construction of a narrow remote point by Baldwin and Smith in [1986].

Non-trivial cut points that are not remote points were found, assuming CH, by Zhu in [1991b].

The consistency of the non-existence of narrow remote points was established by Zhu in [19∞b]. Laver's model for the Borel Conjecture is from Laver [1976].

Theorem 8.4 is a special case of the main theorem of van Douwen [1981a] and Chae and Smith [1980] which states that every space with a countable π-base has a remote point.

9. Mapping a remote point to a near point

In this section we shall see how, in certain situations, one may map a remote point of $\mathbb{H}^*$ to a non-remote point. It should be pointed out that this is a major feat: the homeomorphism that accomplishes this can not be of the form βf for some autohomeomorphism of $\mathbb{H}$. It is generally quite hard, if not impossible, to construct such *non-trivial* homeomorphisms of Čech-Stone remainders.

The idea is quite simple: we take a point u of ω^*, take two cut points of $\mathbb{I}_u$ and map one to the other by an autohomeomorphism of $\mathbb{M}^*$. Once this is done, use the quotient mapping of Theorem 2.4 to turn this autohomeomorphism into one of $\mathbb{H}^*$.

Of course this is easier said than done and a lot of care will have to go into the choice of u and the cut points x and y of $\mathbb{I}_u$ for this to have any chance of success.

The point u of ω^* will be a P-point of character ω_1 and the cut points x and y will have character ω_1 as well. We recall that u is a P-point if it satisfies the following condition: if $\langle U_n \rangle_n$ is a sequence of elements of u then there is $U \in u$ such that U is *almost contained* in every U_n, which means that $U \setminus U_n$ is finite for every n. We write $A \subseteq^* B$ to denote that A is almost contained in B.

Since u is a P-point of character ω_1 we can find a base $\langle U_\alpha : \alpha < \omega_1 \rangle$ for u such that $U_\alpha \subseteq^* U_\beta$ whenever $\beta < \alpha < \omega_1$.

Since the character of x is ω_1, the cofinality of $[0_u, x)$ and the coinitiality of $(x, 1_u]$ are both ω_1 (use Proposition 2.12 to see that these numbers are uncountable). Therefore we can find sequences $\langle a_\alpha : \alpha < \omega_1 \rangle$ in $A_{\{x\}}$ and $\langle b_\alpha : \alpha < \omega_1 \rangle$ in $B_{\{x\}}$, such that $a_{\beta,u} <_u a_{\alpha,u} <_u b_{\alpha,u} <_u b_{\beta,u}$ if $\beta < \alpha < \omega_1$ and $\{x\} = \bigcap_\alpha [a_{\alpha,u}, b_{\alpha,u}]$.

Of course similar sequences $\langle c_\alpha : \alpha < \omega_1 \rangle$ and $\langle d_\alpha : \alpha < \omega_1 \rangle$ can be found for the point y.

We can, upon shrinking the sets U_α somewhat, assume that $U_{\alpha+1}$ is always a subset of U_α (rather than almost a subset) and that

$$a_\beta(n) < a_\alpha(n) < b_\alpha(n) < b_\beta(n) \text{ and } c_\beta(n) < c_\alpha(n) < d_\alpha(n) < d_\beta(n)$$

for all but finitely many $n \in U_\alpha$ whenever $\beta < \alpha$. If $\alpha = \beta + 1$ we can even assume that these inequalities hold for all $n \in U_\alpha$. We shall write

$$F_\alpha = \mathrm{cl}_{\beta\mathbb{M}}\, F(a_\alpha, b_\alpha, U_\alpha) \cap \mathbb{M}^*,$$

and

$$G_\alpha = \mathrm{cl}_{\beta\mathbb{M}}\, F(c_\alpha, d_\alpha, U_\alpha) \cap \mathbb{M}^*$$

for every α.

9.1. LEMMA. *The family $\{F_\alpha : \alpha < \omega_1\}$ is a local base for $\mathbb{M}^*$ at x and so is $\{G_\alpha : \alpha < \omega_1\}$ at y.*

□ Let O be open in $\beta\mathbb{M}$ such that $x \in O$. By Lemma 6.1 we can find $U \in u$, $a \in A_{\{x\}}$ and $b \in B_{\{x\}}$ such that $Cl_{\beta\mathbb{M}}F(a, b, U) \subseteq O$.

Find α such that $U_\alpha \subseteq^* U$ and $a(n) < a_\alpha(n) < b_\alpha(n) < b(n)$ for all $n \in U_\alpha$. From this it follows immediately that $F_\alpha \subseteq O \cap \mathbb{M}^*$.

To finish the proof we should show that every F_α is a neighbourhood of x in $\mathbb{M}^*$. However, one readily verifies that $F_{\alpha+1} \cap (\mathbb{M}^* \setminus F_\alpha) = \varnothing$, so that $x \in \operatorname{Int} F_\alpha$ for all α. □

Now we are ready for the construction of the autohomeomorphism of $\mathbb{M}^*$ that maps x to y.

We shall construct a sequence $\langle h_\alpha : \alpha < \omega_1 \rangle$ of autohomeomorphisms of $\beta\mathbb{M}$ such that the following conditions are satisfied:

1. For every $n \in \mathbb{N}$ and every α we have $h_\alpha(n, 0) = \langle n, 0 \rangle$ and $h_\alpha(n, 1) = \langle n, 1 \rangle$, hence $h_\alpha(0_u) = 0_u$ and $h_\alpha(1_u) = 1_u$ for every $u \in \omega^*$,

2. for every α we have $h_\alpha[\mathbb{M}^* \setminus F_\alpha] = \mathbb{M}^* \setminus G_\alpha$, and
3. if $\beta < \alpha < \omega_1$ then $h_\alpha \restriction (\mathbb{M}^* \setminus F_\beta) = h_\beta \restriction (\mathbb{M}^* \setminus F_\beta)$.

Once we have this sequence we can define $h : \mathbb{M}^* \to \mathbb{M}^*$ by combining the maps $h_\alpha \restriction (\mathbb{M}^* \setminus F_\alpha)$ and sending x to y. It is clear that h is one-to-one and onto. That h is a homeomorphism follows because it is a homeomorphism on every set $\mathbb{M}^* \setminus F_\alpha$ and because it maps F_α onto G_α for every α.

The construction of the h_α will be by induction. There is no loss of generality in assuming that $U_0 = \omega$, that a_0 and c_0 are identically zero and that b_0 and d_0 are identically one.

We shall construct the h_α on $\mathbb{M}$ of course and let Čech and Stone do the rest. Our demands on the h_α are as follows:

1. The map h_α is piecewise linear and monotone on every $\mathbb{I}_n$ and if $n \in U_\alpha$ then $h_\alpha\bigl(a_\alpha(n)\bigr) = c_\alpha(n)$ and $h_\alpha\bigl(b_\alpha(n)\bigr) = d_\alpha(n)$, and
2. if $\beta < \alpha < \omega_1$ then for all but finitely many $n \in U_\beta$ the functions h_α and h_β agree on the intervals $[0, a_\beta(n)]$ and $[b_\beta(n), 1]$.

It should be clear that these conditions are sufficient.

To start the induction we let h_0 be the identity. Now assume that we have constructed h_γ for $\gamma < \alpha$ and that all the demands are met for $\delta < \gamma < \alpha$.

In the successor case, say $\alpha = \beta + 1$, we know that $a_\beta(n) < a_\alpha(n) < b_\alpha(n) < \beta(n)$ and $c_\beta(n) < c_\alpha(n) < d_\alpha(n) < d_\beta(n)$ for all $n \in U_\alpha$. We let h_α agree with h_β on the $\mathbb{I}_n$ with $n \notin U_\alpha$. If $n \in U_\alpha$ we let h_α agree with h_β on $[0, a_\beta(n)] \cup [b_\beta(n), 1]$ but on $[a_\beta(n), b_\beta(n)]$ we make sure that $h_\alpha\bigl(a_\alpha(n)\bigr) = c_\alpha(n)$ and $h_\alpha\bigl(b_\alpha(n)\bigr) = d_\alpha(n)$. A straightforward check will show that h_α is as required.

If α is a limit we let $\langle \alpha_i : i \in \omega \rangle$ be a strictly increasing sequence of ordinals that converges to α. We assume that $\alpha_0 = 0$. From the way we chose the U_α we know that for every i we have

$$a_{\alpha_i}(n) < a_\alpha(n) < b_\alpha(n) < b_{\alpha_i}(n) \quad \text{and} \quad c_{\alpha_i}(n) < c_\alpha(n) < d_\alpha(n) < d_{\alpha_i}(n) \qquad (*)$$

for all but finitely many $n \in U_\alpha$. Choose a strictly increasing sequence $\langle n_i \rangle_i$ of natural numbers with $n_0 = 0$ such that for every i: if $n \in U_\alpha$ and $n \geq n_i$ then $n \in U_{\alpha_i}$ and the inequalities $(*)$ hold.

Now we define h_α. If $n \notin U_\alpha$ and $n \in U_{\alpha_i} \setminus U_{\alpha_{i+1}}$ then h_α agrees with h_{α_i} on $\mathbb{I}_n$. If $n \in U_\alpha$ and $n_i \leq n < n_{i+1}$ then we let h_α agree with h_{α_i} on $[0, a_{\alpha_i}(n)] \cup [b_{\alpha_i}(n), 1]$ but on $[a_{\alpha_i}(n), b_{\alpha_i}(n)]$ we make sure that $h_\alpha\bigl(a_\alpha(n)\bigr) = c_\alpha(n)$ and $h_\alpha\bigl(b_\alpha(n)\bigr) = d_\alpha(n)$. It is again straightforward to check that this h_α is as required.

It now remains to show that this situation can actually occur.

One possibility is to assume the Continuum Hypothesis and do the proof of Theorem 8.2 with a bit of extra care so as to make the ultrafilter u a P-point. Theorem 8.3 will then give us two non-trivial cut points of $\mathbb{I}_u$: one remote, the other non-remote. We conclude that we can map a remote point to different kinds of non-remote points: far and near.

Another possibility is to turn the proof of Theorem 8.2 into an iterated forcing construction.

We shall try to be brief. For a point x of $(\omega\times\mathbb{C})^*$ we can consider the following poset $\mathbb{P}_x$. Its elements are pairs $\langle F,f\rangle$, where F is a finite subset of x and f is a finite partial function from ω to $\mathbb{S}$. Such an f determines a clopen subset of $\omega\times\mathbb{C}$:

$$C_f=\bigcup_{n\in\mathrm{dom}(f)}\{n\}\times B_{f(n)}.$$

We order $\mathbb{P}_x$ by

$$\langle F,f\rangle\le\langle G,g\rangle \qquad\text{iff}\qquad F\supseteq G,\ f\supseteq g \text{ and } C_f\setminus C_g\subseteq\bigcap G.$$

This defines a ccc poset. Indeed, two elements with the same second coordinates are compatible: $\langle F\cup G,f\rangle\le\langle F,f\rangle,\langle G,f\rangle$.

Observe that $\mathbb{P}_x$ is not very interesting if x is not remote: as soon as F has a nowhere dense element, no element $\langle F,f\rangle$ can be extended by enlarging the coordinate f.

Now assume that x is remote. Then the following types of sets are dense in $\mathbb{P}_x$:

$$D_P=\bigl\{\langle F,f\rangle : P\in F\bigr\},$$

where $P\in x$ and

$$E_n=\bigl\{\langle F,f\rangle : \text{ there is an } m\ge n \text{ such that } m\in\mathrm{dom}(f)\bigr\},$$

where $n\in\omega$. For the D_P observe that always $\bigl\langle F\cup\{P\},f\bigr\rangle\le\langle F,f\rangle$. For the E_n we use that for any $\langle F,f\rangle$ the set $\bigcap F$ meets infinitely many of the sets $\{m\}\times\mathbb{C}$ in a set with nonempty interior.

Now let G be generic on $\mathbb{P}_x$ and set

$$f_G=\bigcup\Bigl\{f:\exists F\bigl[\langle F,f\rangle\in G\bigr]\Bigr\}.$$

The set $X_G=S(\mathrm{dom}(f_G),f_G)$ is a noncompact clopen set and it is such that $X_G\setminus P$ is compact for every $P\in x$. Also note that $\mathrm{dom}(f_G)$ is almost contained in every element of the ultrafilter $u=\{U:U\times\mathbb{C}\in x\}$.

Start with any model M of ZFC and set up a finite support iterated forcing construction $\langle\mathbb{P}_\alpha:\alpha<\omega_1\rangle$ as follows. At every stage we let $\dot{x}_\alpha$ be a $\mathbb{P}_\alpha$-name for a remote point and we let $\mathbb{P}_{\alpha+1}=\mathbb{P}_\alpha*\mathbb{P}_{\dot{x}_\alpha}$. The set added by $\mathbb{P}_{\dot{x}_\alpha}$ will be denoted by X_α.

The remote point x_α is chosen in such a way that $X_\beta\in x_\alpha$ for every $\beta<\alpha$. It is not too hard to show that this can always be done (use Theorem 8.4).

In the end we get a point x in $(\omega\times\mathbb{C})^*$ generated by the family $\{X_\alpha:\alpha<\omega_1\}$ and its associated ultrafilter u on ω. The point u is a P-point of ω^* and x is a remote point of $\omega\times\mathbb{C}$ that is just like the remote point from Theorem 8.2. To it we may apply the proof of Theorem 8.3.

In this way we get the consistency with $\neg$CH of an autohomeomorphism of $\mathbb{H}^*$ that moves a remote point to a non-remote point.

Notes for Section 9.

The results from this section are due to Yu [1991]; except for the iterated-forcing construction, this is an elaboration of Exercise VIII A10 from Kunen [1980]. To the best of my knowledge this is the first non-trivial homeomorphism of $\mathbb{H}^*$.

In [1980] van Mill and Mills exhibited, assuming CH, a remote point x of $\mathbb{H}$ such that $h(x)$ is remote for every autohomeomorphism h of $\mathbb{H}^*$.

10. The number of subcontinua of $\mathbb{H}^*$

In this section we make a beginning with the topological classification of the proper subcontinua of $\mathbb{H}^*$. It will become clear that our knowledge is still quite limited.

We discuss the known ZFC results first; we will find nine different continua. In fact, we show that every standard subcontinuum contains at least eight topologically different subcontinua. This suggests obvious questions, we will save these for Section 13.

Let $\mathbb{I}_u$ be any standard subcontinuum.

10.1. Lemma. *Let $\langle a_n\rangle_n$ and $\langle b_n\rangle_n$ be sequences in P_u such that*

$$a_n \le_u a_{n+1} <_u b_{n+1} \le_u b_n$$

for all n. Then there is $x \in P_u$ such that $a_n <_u x <_u b_n$ for all n.

□ Choose a decreasing sequence $\langle U_n\rangle_n$ of elements of u with empty intersection such that satisfying $U_0 = \omega$ and

$$a_n(i) \le a_{n+1}(i) < b_{n+1}(i) \le b_n(i), \qquad (i \in U_{n+1}).$$

Next define x by $x(i) = \frac{1}{2}\bigl(a_n(i) + b_n(i)\bigr)$ for $i \in U_n \setminus U_{n+1}$. □

Now we are ready to define the eight subcontinua of $\mathbb{I}_u$. There will be six intervals and two indecomposable continua.

To begin we let $K_1 = \mathbb{I}_u$. Next we take a strictly increasing sequence $\langle a_n\rangle_n$ in P_u with limit layer L_1 (cf. Proposition 2.12) and we set $K_2 = [0_u, L_1]$ and $K_3 = [L_1, 1_u]$. Take another strictly increasing sequence $\langle b_n\rangle_n$ in P_u, this time above L_1 and let L_2 be its limit layer; put $K_4 = [L_1, L_2]$. Choose yet another sequence $\langle c_n\rangle_n$ in $P_u \cap [L_1, L_2]$, strictly decreasing with limit layer L_3. We set $K_5 = [L_3, L_2]$ and $K_6 = [L_1, L_3]$. Finally we let $K_7 = \{0_u\}$ and $K_8 = L_1$.

It is clear that K_7 is different from K_8 and that both are different from K_1 through K_6.

To distinguish the first six continua we observe that any homeomorphism between them should map layers to layers because layers are indecomposable. Furthermore we note that each K_i has two distinguished end layers: the only layers that, when removed, leave K_i connected. These end layers should therefore be mapped to end layers.

We inspect the end layers: K_1 has two one-point end layers and K_2 and K_3 each have one but in K_2 the other end layer is a G_δ-set and in K_3 it is not—by Lemma 10.1. We can distinguish K_4, K_5 and K_6 by the number of end layers that are G_δ-sets.

To find our ninth continuum we have to do some more work. The continuum K_9 will be indecomposable and non-degenerate. To distinguish it from K_8 we negate the following property that K_8 has: *every nonempty G_δ subset has nonempty interior, or using complements: no proper F_σ subset is dense*. To see that K_8 has this property, we note that the space $[0_u, L_1)$ is σ-compact and locally compact and that, because $\mathbb{M}^*$ is an F-space, we have $[0_u, L_1] = \beta[0_u, L_1)$.

We shall construct a strictly increasing sequence $\langle M_n \rangle_n$ of indecomposable continua and let $K_9 = \overline{\bigcup_n M_n}$. Clearly K_9 is a continuum: $\bigcup_n M_n$ is connected. Since M_n is nowhere dense in M_{n+1} for every n, the set $\bigcup_n M_n$ is a proper dense F_σ-subset of K_9. To see that K_9 is indecomposable, consider a proper subcontinuum K. If K is disjoint from every M_n then K is nowhere dense in K_9, because it misses a dense subset. On the other hand, if it meets some M_n then it must be contained in one of them, because we have $M_k \subseteq K$ or $K \subseteq M_k$ for every $k \geq n$ and $\bigcup_k M_k \subseteq K$ is impossible.

For the construction of the continua M_n we need the notion of a Q-point. A point u of ω^* is said to be a *Q-point* if for every finite-to-one function $f : \omega \to \omega$ there is an element of u on which f is one-to-one. It is not too hard to show that it suffices to consider monotone functions only.

It is easy to find non-Q-points. Define for example π by $\pi(k) = n$ iff $2^n \leq k < 2^{n+1}$ (and $\pi(0) = 0$). It is easily seen that the family

$$\{\omega \setminus A : \pi \text{ is one-to-one on } A\}$$

has the finite intersection property and that no ultrafilter extending it is a Q-point.

Using Lemma 6.2 it is straightforward to prove the following lemma.

10.2. LEMMA. *Let $[a_u, b_u]$ be a standard subcontinuum. Then $[a_u, b_u]$ is contained in an indecomposable (proper) subcontinuum of $\mathbb{H}^*$ if and only if u is not a Q-point.*

□ Suppose first u is a Q-point and let $[c_v, d_v]$ be any standard subcontinuum containing $[a_u, b_u]$. We may assume that for every n there is an m such that $[a_n, b_n] \subseteq [c_m, d_m]$. This defines a finite-to-one map φ from ω to ω. Fix $U \in u$ on which φ is one-to-one. It follows that $f(u) = v$ and that $[a_u, b_u]$ is a interval of $[c_v, d_v]$.

We conclude that $[a_u, b_u]$ is not contained in the layer of any other standard subcontinuum. Since indecomposable subcontinua must be contained in the layer of some standard subcontinuum we see that $[a_u, d_u]$ is not contained in any indecomposable subcontinuum of $\mathbb{H}^*$.

Conversely suppose that u is not a Q-point and fix an increasing finite-to-one map φ that is not one-to-one on any element of u. We may assume that φ is surjective. For every n let $[c_n, d_n]$ be the smallest interval that covers all intervals $[a_i, b_i]$ with $\varphi(i) = n$. Finally let $v = \varphi(u)$.

Now Lemma 6.2 applies and we can conclude that $[a_u, b_u]$ is contained in a layer of $[c_v, d_v]$. □

Our aim is to construct a sequence $\langle u_n \rangle_n$ in ω^* such that $\pi(u_n) = u_{n+1}$ for all n and π is not one-to-one on any element of any u_n. Once we have this sequence we start with a standard subcontinuum $[a_{u_0}, b_{u_0}]$ and inductively apply the proof of Lemma 10.2 to find standard subcontinua $[a_{u_n}, b_{u_n}]$ such that for every n the continuum $[a_{u_n}, b_{u_n}]$ is contained in a layer of $[a_{u_{n+1}}, b_{u_{n+1}}]$, call this layer M_n. This gives us our sequence $\langle M_n \rangle_n$.

For the construction of $\langle u_n \rangle_n$ we consider the family

$$\mathcal{F} = \Bigl\{ F \subseteq \omega : \exists n \bigl[\pi \text{ is one-to-one on } \pi^n[\omega \setminus F] \bigr] \Bigr\}.$$

It is not too hard to verify that $\mathcal{F} \cup \bigl\{ [n, \omega) : n \in \omega \bigr\}$ has the finite intersection property. Now let u_0 be any ultrafilter that extends $\mathcal{F}$ and set $u_n = \pi^n(u_0)$ for $n > 0$. It should be clear that $\langle u_n \rangle_n$ is as required.

If one is willing to go beyond ZFC a bit more can be said. First we quote a theorem due to Dow. If κ and λ are regular cardinals then a $\langle \kappa, \lambda \rangle$-gap in an ordered set is a pair of sequences $S = \langle x_\alpha : \alpha < \kappa \rangle$, $T = \langle y_\alpha : \alpha < \lambda \rangle$ such that S is increasing, T is decreasing, S is below T and there is no element x such that $x_\alpha < x < y_\beta$ for all $\alpha < \kappa$ and $\beta < \lambda$.

10.3. Theorem (Dow). *If $\kappa \leq \mathfrak{c}$ is a regular uncountable cardinal then there is an ultrafilter u_κ such that P_{u_κ} has a $\langle \omega, \kappa \rangle$-gap, but no $\langle \omega, \lambda \rangle$-gap for any $\lambda < \kappa$.*

Using this theorem we can find nonhomeomorphic standard subcontinua of $\mathbb{H}^*$, one for every regular uncountable cardinal that is not larger than $\mathfrak{c}$: if $\kappa < \lambda$ then a layer in $\mathbb{I}_{u_\kappa}$ that corresponds to an $\langle \omega, \kappa \rangle$-gap can not be mapped to any layer of $\mathbb{I}_{u_\lambda}$. For this to be of any use we must assume $\neg$CH of course.

To find different indecomposable subcontinua of $\mathbb{H}^*$ one can use the following theorem.

10.4. Theorem (Zhu). *Assume that $\mathfrak{c}$ is regular and satisfies $2^{<\mathfrak{c}} = \mathfrak{c}$, and let κ be a regular uncountable cardinal less than or equal to $\mathfrak{c}$. If one adds κ Cohen reals then in the resulting model there is, in some $\mathbb{I}_u$, a layer in which the intersection of fewer than κ open sets has nonempty interior and in which there is also a point of character κ.*

Notes for Section 10.

Lemma 10.1 is well-known; it is a basic fact about ultrapowers of $\mathbb{R}$.

The continua K_1 through K_8 were found by Smith in [1986]. In [1977] van Douwen announced five different subcontinua: two indecomposable (K_8 and K_9) and three decomposable, distinguished by the number of one-point end layers. In his notes van Douwen also considered the number of G_δ-sets among the end layers. The construction of K_9 given here is due to Zhu [19∞a].

The existence of Q-points is independent of ZFC: on the one hand it is straightforward to construct a Q-point assuming the Continuum Hypothesis; on the other hand the principle NCF (see Section 11) implies that no Q-points exist.

Theorem 10.3 was proved by Dow in [1984].

11. Composants and NCF

Since $\mathbb{H}^*$ is an indecomposable continuum it becomes interesting to study its composants.

11.1. Definition. Let X be an indecomposable continuum. The relation 'x and y are contained in a proper subcontinuum of X' is an equivalence relation on X. The equivalence classes under this relation are called the *composants* of X.

The relation above is clearly reflexive and symmetric; indecomposability guarantees that it is also transitive. An obvious question is what the number of composants of an indecomposable continuum can be. In the metric case this number is always $\mathfrak{c}$ and in the non-metric case there are examples of continua with one, two or 2^κ composants for any infinite κ.

We shall see that the number of composants of $\mathbb{H}^*$ is determined completely by a combinatorial property of the set ω^*.

11.2. Theorem. *Every composant of* $\mathbb{H}^*$ *contains a point of* ω^*.

□ Consider the quotient map q from $\mathbb{M}^*$ onto $\mathbb{H}^*$ from Theorem 2.4. There is a point $u \in \omega^*$ such that $x \in q[\mathbb{I}_u]$. The standard subcontinuum $q[\mathbb{I}_u]$ connects x and u. □

By this theorem we can concentrate on the composants of the points of ω^*. To be able to characterize when two points of ω^* are in the same composant of $\mathbb{H}^*$ we make the following definition.

11.3. Definition. Two points u and v of ω^* are said to be *nearly coherent* if there are finite-to-one maps f and g from ω to ω such that $f(u) = g(v)$.

This notion has proved itself useful in various circumstances. The principle NCF (Near Coherence of Filters), which says that any two points of ω^* are nearly coherent, implies that many objects have a simple structure. It is not too hard to show that u and v are nearly coherent iff there is one non-decreasing surjection f such that $f(u) = f(v)$.

The following theorem shows the connection between NCF and the composants of $\mathbb{H}^*$.

11.4. Theorem. *Let* $u, v \in \omega^*$. *Then* u *and* v *are in the same composant of* $\mathbb{H}^*$ *if and only if they are nearly coherent.*

□ For the first implication assume that u and v are contained in some standard subcontinuum K of $\mathbb{H}^*$. Take sequences $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ in $\mathbb{H}$ such that $a_n < b_n < a_{n+1}$ for all n, and an ultrafilter w such that $K = [a_w, b_w]$. We may assume that ω is covered by the intervals $[a_n, b_n]$. Now define $f : \omega \to \omega$ by $f(i) = n$ iff $i \in [a_n, b_n]$. This map is finite-to-one and it maps u and v to w.

To prove the converse we let f be finite-to-one and non-decreasing such that $f(u) = f(v) = w$ for some $w \in \omega^*$. We let I_n be the smallest interval containing $f^{\leftarrow}(n)$. It should now be clear that $u, v \in I_w$ and hence that u and v are in the same composant. □

This theorem implies that the number of composants of $\mathbb{H}^*$ is equal to the number of equivalence classes of ω^* under the relation of near coherence (that this is indeed an equivalence relation follows from the theorem).

This number depends on extra axioms of set theory. On the one hand CH implies that there are $2^{\mathfrak{c}}$ equivalence classes and on the other hand the principle NCF is consistent with ZFC. We see that the number of composants of $\mathbb{H}^*$ can not be determined in ZFC.

We finish this section with an interesting application of the results presented in this paper to the space ω^*

The problem is to cover ω^* with 'small' P-sets. A P-set in a topological space is one with the property that the intersection of countably many of its neighbourhoods is again a neighbourhood of it. A P-point is a point x such that $\{x\}$ is a P-set. For points of ω^* we now have two notions of P-point; see Section 9. It is well-known that these notions are the same. It is easy to see that a compact space in which every point is a P-point must be finite, hence not every point of ω^* is a P-point.

One may therefore wonder whether compact spaces may be covered with 'small' P-sets. We take small to mean closed and nowhere dense. It turns out that under CH the space ω^* can not be covered with closed nowhere dense P-sets. On the other hand, a well-known consequence of NCF is that ω^* can be covered with nowhere dense P-sets. This can be seen as follows: NCF implies that for every $u \in \omega^*$ there is a finite-to-one map $\varphi : \omega \to \omega$ such that $v = \varphi(u)$ is a P-point. It is readily seen that the set $\beta\varphi^{\leftarrow}(v)$ is a closed nowhere dense P-set of ω^*.

Using some of the results on subcontinua of $\mathbb{H}^*$ one can prove the following strengthening of this observation.

11.5. Theorem (Zhu). NCF *implies that ω^* can be covered by a chain of nowhere dense P-sets.*

The proof of this theorem relies on the following lemma.

11.6. Lemma. *Let u be a P-point of ω^* and let $\langle [a_n, b_n] \rangle_n$ be a discrete sequence of intervals in $\mathbb{H}$. Then $\omega^* \cap [a_u, b_u]$ is a nowhere dense P-set of ω^*.*

$\square$ We assume that $a_n < b_n < a_{n+1}$ for all n and define $\varphi : \omega \to \omega$ by $\varphi(i) = \min\{n : i \le b_n\}$. Then $\omega^* \cap [a_u, b_u] = A^* \cap \beta\varphi^{\leftarrow}(u)$, where $A = \omega \cap \bigcup_n [a_n, b_n]$. $\square$

Proof of Theorem 11.5. We construct a sequence $\langle u_\alpha \rangle_\alpha$ of P-points and corresponding standard subcontinua K_α such that $K_\alpha \subset K_\beta$ whenever $\alpha < \beta$ and such that $\mathbb{H}^* = \bigcup_\alpha K_\alpha$.

By NCF we may pick a P-point u_0. We let K_0 be determined by u_0 and $\langle [n, n + 1/2] \rangle_n$.

At a successor stage, given u_α and K_α, we apply the proof of Lemma 10.2 (and NCF) to find a finite-to-one map φ_α, a P-point $u_{\alpha+1}$ and a standard subcontinuum $K_{\alpha+1}$ such that K_α is contained in a layer L_α of $K_{\alpha+1}$ and $u_{\alpha+1} = \varphi_\alpha(u_\alpha)$.

If α is a limit consider the union $\bigcup_{\gamma<\alpha} K_\gamma = \bigcup_{\gamma<\alpha} L_\gamma$. If this union equals $\mathbb{H}^*$ then stop, otherwise pick a point $x \in \mathbb{H}^*$ that is not in the union. By NCF ($\mathbb{H}^*$ has one composant) we may find a standard subcontinuum K_α, determined by a P-point u_α, containing both u_0 and x. Now apply Theorem 5.9 to conclude that $\bigcup_{\gamma<\alpha} K_\gamma \subset K_\alpha$ (in fact the closure of the union is indecomposable and hence contained in a layer of K_α).

This construction will stop before the cardinal number $(2^{\mathfrak{c}})^+$ and thus produce our chain $\langle K_\alpha \cap \omega^* \rangle_\alpha$ of nowhere dense P-sets covering ω^*. $\square$

11.7. Remark. Consider the sequence $\langle u_\alpha \rangle_\alpha$ constructed in the proof above. What we have seen is that whenever $\alpha < \beta$ there is a finite-to-one map φ such that $\varphi(u_\alpha) = u_\beta$ (use Lemma 10.2) and for every $u \in \omega^*$ are an α and a finite-to-one map ψ such that $\psi(u) = u_\alpha$.

Let us consider the following ordering on ω^*: say $u \leq v$ iff there is a finite-to-one map φ such that $\varphi(v) = u$ (this is almost the Rudin-Keisler order).

Now by definition NCF says that the ordered set $\langle \omega^*, \leq \rangle$ is downward directed. What we have seen is that it is equivalent to the formally much stronger statement that there is a linearly ordered coinitial subset in this ordering. Indeed, the usual consistency proofs for NCF produce exactly such sets.

11.8. Remark. It may seem strange that the structure of a continuum may have effect on the structure of ω^*, a zero-dimensional space. The proof of Theorem 11.5 uses the structure of $\mathbb{H}^*$ in an essential way: the saving feature at the limit stage is that indecomposable subcontinua are either disjoint or comparable. I don't see how this may be translated into a direct argument that would avoid $\mathbb{H}^*$ altogether.

Notes for Section 11.

In [1927] Mazurkiewicz proved that every metric indecomposable continuum contains a Cantor set K no two points of which lie in the same composant. This more than shows that the number of composants of a metric indecomposable continuum equals $\mathfrak{c}$.

Non-metric indecomposable continua with one or two composants were constructed by Bellamy in [1978] and in [1976] Smith constructed for every infinite κ an indecomposable continuum with 2^κ composants.

In [1970] M. E. Rudin constructed a family of $2^\mathfrak{c}$ points in ω^* such that no two of them are near coherent, and then proved one half of Theorem 11.4 (points that are in the same composant are near coherent). In [1978] Mioduszewski proved the same half and in [1980] essentially announced the converse.

The papers from [1986] and [1987] by Blass survey many applications of NCF. Proofs of its consistency can be found in Blass and Shelah [1987, 1989].

Theorem 11.5 is due to Zhu [1991a]. For a proof that, under CH, the space ω^* can not be covered by nowhere dense P-sets see Kunen, van Mill and Mills [1980]. Another model in which ω^* can be covered by nowhere dense P-sets can be found in Balcar, Frankiewicz and Millls [1980].

12. Miscellanea from van Douwen's notes

In this section I collect some results from van Douwen's notes that do not seem to fit elsewhere.

The first result shows once more that $\mathbb{H}^*$ is indecomposable.

12.1. Theorem. *Let F be a proper closed subset of $\mathbb{H}^*$ with nonempty interior. Then F has a closed subset homeomorphic to ω^* that is a retract of F.*

□ Apply Proposition 3.2 twice, first to F and a point not in F to obtain sequences $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ such that $F \subseteq F(a, b, \omega)^*$ and then to a point in the interior of F and the closed set $\mathrm{cl}_{\mathbb{H}^*}(\mathbb{H}^* \setminus F)$ to find sequences $\langle c_n \rangle_n$ and $\langle d_n \rangle_n$ such that $F(c, d, \omega)^* \subseteq F$. We may assume that every interval $[c_m, d_m]$ is contained in some interval $[a_n, b_n]$.

Define $r : F(a,b,\omega) \to F(c,d,\omega)$ as follows: for every n let m_n be the first m such that $[c_m, d_m] \subseteq [a_n, b_n]$ if such an m exists, otherwise let m_n be minimal subject to $c_m > b_n$. Now map, for every n, the interval $[a_n, b_n]$ to the point c_{m_n}.

The map r is a retraction of $F(a,b,\omega)$ onto $\{c_{m_n} : n \in \omega\}$. Its extension βr is a retraction of $F(a,b,\omega)^*$ onto $\{c_{m_n} : n \in \omega\}^*$; it retracts F onto this set as well. □

We conclude that closed sets with nonempty interior have $2^{\mathfrak{c}}$ components, the maximum number possible.

The second result shows that, although proper subcontinua of $\mathbb{H}^*$ are nowhere dense, they may stretch out over a long distance.

12.2. Theorem. *Let $\{U_n : n \in \omega\}$ be a sequence of nonempty open subsets of $\mathbb{H}^*$. Then there is a proper indecomposable subcontinuum of $\mathbb{H}^*$ that meets every set U_n.*

□ For each n choose a discrete subset D_n of $\mathbb{H}^*$ such that $D^* \subseteq U_n$. Next find a discrete sequence $\langle I_n : n \in \omega \rangle$ of closed and nondegenerate intervals such that $I_n \cap D_m \neq \varnothing$ whenever $n \geq m$. It follows immediately that every standard subcontinuum I_u intersects every U_n.

By Lemma 10.2 any non Q-point u will provide us with an indecomposable continuum that meets every U_n. □

13. Some questions

In this section we collect some questions that are suggested by the results presented in this survey.

13.1. Question (Van Douwen). *Is there in some $\mathbb{I}_u$ a non-trivial cut point?*

This question needs no real motivation; once one identifies the obvious cut points one wonders whether there are more of them. By Lemma 7.2 this question asks whether there is a weak cut point of $\mathbb{H}^*$ that is not a near point. We note that Theorem 8.3 provides a conditional positive answer. On the other hand, none of the results in this paper say something about the other end of the spectrum.

13.2. Question. *Is there an ultrafilter u such that $\mathbb{I}_u$ has no nontrivial cut points?*

In Section 10 we discovered nine topologically different subcontinua of $\mathbb{H}^*$. It seems unlikely that this is the best one can say in ZFC.

13.3. Question. *What is the number of topologically different subcontinua of $\mathbb{H}^*$?*

The 'right' answer to this question should be: $2^{\mathfrak{c}}$. We remark that the remainder of $\mathbb{R}^2$ does indeed have $2^{\mathfrak{c}}$ different subcontinua. This was established by Browner Winslow in [1980] and van Douwen in [1981b].

The number of subcontinua can be at least $\mathfrak{c}$; if one adds enough Cohen reals then one can find $\mathfrak{c}$ different continua in $\mathbb{H}^*$. This follows from Theorems 10.3 and 10.4. Theorem 10.3 gives better information: there are always at least as many standard subcontinua of $\mathbb{H}^*$ as there are regular cardinals below (or equal to) $\mathfrak{c}$; it also gives a positive answer, under $\neg$CH, to the following question; one would like a ZFC result of course.

13.4. Question. *Are there u and v in ω^* such that $\mathbb{I}_u$ and $\mathbb{I}_v$ are nonhomeomorphic.?*

Let us note however that the Continuum Hypothesis implies that for any two ultrafilters u and v the sets P_u and P_v are isomorphic as ordered sets. The point is that these sets satisfy Lemma 10.1, i.e., they are η_1-sets. It is an old result of Hausdorff from [1914] that any two η_1-sets of cardinality ω_1 are isomorphic.

In connection with the result of Yu, Section 9, the following questions come to mind.

13.5. Question. *Is there, in* ZFC*, a non-trivial homeomorphism of $\mathbb{H}^*$?*

For the space ω^* the answer is negative, see Shelah and Steprāns [1988]. Should this question have a positive answer then the following question becomes interesting as well:

13.6. Question. *Determine whether the 'real' remote points of van Douwen (Theorem 8.4) can be mapped to non remote points by an autohomeomorphism of $\mathbb{H}^*$.*

As a first try one may consider large points. There are two reasons to do this: (i) both kinds of points have fairly concrete descriptions and (ii) neither kind of point is a weak cut point (so there is no obvious reason why large points can't be mapped to remote points).

It follows from Lemma 6.1 that the decomposition into layers is upper semi-continuous on the whole space $\mathbb{M}^*$. It induces, via the quotient map of Lemma 2.4, an upper semi-continuous decomposition of $\mathbb{H}^*$. The quotient of $\mathbb{H}^*$ obtained in this way looks a lot like a solenoid. It would be interesting to investigate the structure of this space, for example its dynamical properties. This would probably necessitate an investigation of the ordered continua X_u defined right after Corollary 2.10.

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