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ELEMENTARITY AND DIMENSIONS

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Abstract. We give an alternative proof of Fedorchuk's recent result that $\dim X \leqslant \operatorname{Dg} X$ for compact Hausdorff spaces X. We use the Löwenheim-Skolem theorem to reduce the problem to the metric case.

1. Introduction

From the various *topological* notions of dimension that have been proposed the best-known and most widely used are ind and Ind, the small and large inductive dimension, and dim, the covering dimension. These capture in various ways the intuition behind dimension. The inductive dimensions formalize the idea that "a line is separated by points, a surface by lines and space by surfaces", whereas dim captures dimension as "number of directions", especially through the theorem on partitions [2, 7.2.15]. These functions assume the same values for all separable metrizable spaces and assign the correct dimension to Euclidean n-space.

In [1] Brouwer proposed another notion of dimension, Dimensionsgrad (Dg), based on cuts. It was established only recently in [4] that Dg coincides with the familiar dimension functions on the class of (locally) compact metric spaces. Outside of this class Dg and the dimension functions diverge: there is, for each n, a locally connected complete separable metric space X_n with $\operatorname{Dg} X_n = 1$ and $\dim X_n = n$, see [5].

Recently Fedorchuk proved that $\dim X \leqslant \operatorname{Dg} X$ for compact Hausdorff spaces X. The purpose of this note is to reprove this and Vedenissof's inequality $\dim X \leqslant \operatorname{Ind} X$ (for normal spaces) by model-theoretic means.

The arguments in this paper seem to indicate that Dg is somewhat more complex than the common dimension functions, which may help to explain why Fedorchuk's proof of his inequality is so much more involved than the fairly straightforward proof of Vedenissof's inequality.

2. Preliminaries

2.1. **Dimensions.** We repeat the definitions of covering dimension and large inductive dimension. We say that covering dimension of a normal space X is at most n, in symbols $\dim X \leqslant n$, if every finite open cover has a refinement of order at most $n+1$ (i.e., no point is in more than $n+1$ members of the refinement). As usual $\dim X$ is defined to be the minimum n for which this holds (or ∞ if there is no such n).

The large inductive dimension is defined by recursion: $\operatorname{Ind} X \leqslant n$ means that between every two disjoint closed sets A and B there is a *partition* C with $\operatorname{Ind} C \leqslant$

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$n-1$, where C is a partition between A and B if $X \setminus C$ can be written as the union of two disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$. This recursion starts with $\operatorname{Ind} X \leqslant -1$ iff $X = \emptyset$.

The Dimensionsgrad is defined similarly but now C should be a *cut* between A and B, which means that it is closed and meets every continuum that intersects both A and B.

2.2. **Lattices.** In [10] Wallman showed that to every distributive lattice L with $\mathbf{0}$ and $\mathbf{1}$ one can associate a compact T_1-space wL, its *Wallman representation*, with a base for the closed sets that is a homomorphic image of L. The underlying set of wL is the set of all ultrafilters on L and for every element a of L the set $\bar{a} = \{u \in wL : a \in u\}$ is a basic closed set in wL. The homomorphism $a \mapsto \bar{a}$ is one-to-one if and only if L is *separative*, which means that whenever $a \nleqslant b$ there is $c \leqslant a$ with $c > \mathbf{0}$ and $c \sqcap b = \mathbf{0}$. The space wL is Hausdorff if and only if L is *normal*, which means that whenever $a \sqcap b = \mathbf{0}$ there are f and g with $a \sqcap f = \mathbf{0}$, $b \sqcap g = \mathbf{0}$ and $f \sqcup g = \mathbf{1}$.

Unlike the Stone representation for Boolean algebras the Wallman representation is not one-to-one. Certainly every compact T_1-space X is the representation of its own lattice of closed sets, which we denote by 2^X, but one also has $X = w\mathcal{B}$ whenever $\mathcal{B}$ is a base for the closed sets of X that is closed under finite unions and intersections. Thus, e.g., the unit interval is also the Wallman representation of the family of finite unions of closed intervals with rational end points.

2.3. **Elementary sublattices.** Our proofs of Fedorchuk's and Vedenissof's inequalities involve elementary sublattices of 2^X. A sublattice L of 2^X is an *elementary* sublattice if every equation with parameters from L that has a solution in 2^X already has a solution in L. Here 'equation' should be taken in a very wide sense. What we demand is: whenever $\phi(x, y, \ldots, a, b, \ldots)$ is a lattice-theoretic formula with its free variables among x, y, … and its parameters a, b, … from L and if there are x, y, … in 2^X such that ϕ holds in 2^X then there are such x, y, … in L.

For example an elementary sublattice of 2^X is automatically separative: if $a \nleqslant b$ in L then $(x \leqslant a) \wedge (x > \mathbf{0}) \wedge (x \sqcap b = \mathbf{0})$ is an equation with parameters — a, b and $\mathbf{0}$ — from L and with a solution in 2^X, hence there must be a $c \in L$ with $(c \leqslant a) \wedge (c > \mathbf{0}) \wedge (c \sqcap b = \mathbf{0})$.

Likewise L must be normal: if $a, b \in L$ and $a \sqcap b = \mathbf{0}$ then the equation $(a \sqcap x) \wedge (b \sqcap y) \wedge (x \sqcup y = \mathbf{1})$ has a solution in 2^X, hence there are f and g in L with $(a \sqcap f) \wedge (b \sqcap g) \wedge (f \sqcup g = \mathbf{1})$.

Below, when proving Fedorchuk's inequality we shall see more complicated equations/formulas that will involve quantifiers and this is where the strength of the notion of elementarity will become apparent.

An important result is the Löwenheim-Skolem theorem, which says, in our context, that given a subfamily $\mathcal{F}$ of 2^X one can always find an elementary sublattice L of 2^X with $\mathcal{F} \subseteq L$ and $|L| \leqslant |\mathcal{F}| \cdot \aleph_0$. This provides an inroad to a strong version of Mardešić's Factorization theorem, see [6, Theorem 5.3] for an example of its use and the thesis [9] for a systematic study of the properties that the factorizing space inherits from the domain. A proof of the Löwenheim-Skolem theorem can be found in [8, Section 3.1].

3. Formulas for dimensions

3.1. **Covering dimension.** We use Hemmingsen's characterization from [7] (see also [2, Corollary 7.2.14]) to make a lattice-theoretic formula that characterizes covering dimension in terms of closed sets. The formula, abbreviated δ_n, is

$$\begin{aligned}(1)\quad &(\forall x_1)(\forall x_2)\cdots(\forall x_{n+2})(\exists y_1)(\exists y_2)\cdots(\exists y_{n+2})\\ &\bigl[(x_1\sqcap x_2\sqcap\cdots\sqcap x_{n+2}=\mathbf{0})\to\bigl((x_1\leqslant y_1)\wedge(x_2\leqslant y_2)\wedge\cdots\wedge(x_{n+2}\leqslant y_{n+2})\\ &\qquad\wedge(y_1\sqcap y_2\sqcap\cdots\sqcap y_{n+2}=\mathbf{0})\wedge(y_1\sqcup y_2\sqcup\cdots\sqcup y_{n+2}=\mathbf{1})\bigr)\bigr].\end{aligned}$$

Hemmingsen's theorem simply says that, for compact spaces, $\dim X\leqslant n$ if and only if the lattice 2^X satisfies the formula δ_n.

A standard shrinking-and-expanding argument will show that for a compact Hausdorff space X one has $\dim X\leqslant n$ if and only if some (every) lattice base for its closed sets satisfies the formula δ_n.

3.2. **Large inductive dimension.** The definition of large inductive dimension can be couched in terms of closed sets quite easily. A partition C between two disjoint closed sets A and B can be described by two closed sets F and G such that $F\cup G=X$, $F\cap A=\emptyset$ and $G\cap B=\emptyset$: the intersection $F\cap G$ is a partition between A and B; thus the following formula $\operatorname{part}(u,x,y,a)$ states that u is a partition between x and y in the (sub)space a:

$$(\exists f)(\exists g)\bigl((x\sqcap f=\mathbf{0})\wedge(y\sqcap g=\mathbf{0})\wedge(f\sqcup g=a)\wedge(f\sqcap g=u)\bigr).$$

This enables us to give a recursive definition of a formula $I_n(a)$ for the large inductive dimension:

$$\begin{aligned}(2)\quad &(\forall x)(\forall y)(\exists u)\\ &\bigl[\bigl(((x\leqslant a)\wedge(y\leqslant a)\wedge(x\sqcap y=\mathbf{0})\bigr)\to\bigl(\operatorname{part}(u,x,y,a)\wedge I_{n-1}(u)\bigr)\bigr];\end{aligned}$$

the recursion starts with $I_{-1}(a)$ abbreviating $a=\mathbf{0}$.

Thus a compact space X satisfies $\operatorname{Ind}X\leqslant n$ if and only if 2^X satisfies $I_n(\mathbf{1})$. More generally, if X has a lattice base $\mathcal{B}$ for its closed sets that satisfies $I_n(\mathbf{1})$ then $\operatorname{Ind}X\leqslant n$; this follows readily by induction, once one realizes that $\{F\in\mathcal{B}:F\subseteq A\}$ is a lattice base for the closed sets in the subspace A, when $A\in\mathcal{B}$.

The converse is not true in that not every lattice base for the closed sets of a space X with $\operatorname{Ind}X\leqslant n$ must satisfy $I_n(\mathbf{1})$. A simple example is given by the unit interval $[0,1]$ and the lattice base generated by the subbase $\bigl\{[0,q]:q\text{ rational}\bigr\}\cup\bigl\{[p,1]:p\text{ irrational}\bigr\}$. This lattice does not satisfy $I_n(\mathbf{1})$ for any n.

3.3. **Dimensionsgrad.** As defined above a cut between two (disjoint) closed sets A and B is a closed set C such that every continuum from the ambient space that intersects A and B also intersects C. If we let $\operatorname{conn}(a)$ abbreviate

$$(\forall x)(\forall y)\bigl[\bigl((x\sqcap y=\mathbf{0})\wedge(x\sqcup y=a)\bigr)\to\bigl((x=\mathbf{0})\vee(x=a)\bigr)\bigr],$$

i.e, "a is connected", and use $\operatorname{cut}(u,x,y,a)$ to denote

$$(\forall v)\bigl[\bigl((v\leqslant a)\wedge\operatorname{conn}(v)\wedge(v\sqcap x\neq\mathbf{0})\wedge(v\sqcap y\neq\mathbf{0})\bigr)\to(v\sqcap u\neq\mathbf{0})\bigr],$$

i.e., "u is a cut between x and y in the (sub)space a", then we get the following recursive definition of a formula $\Delta_n(a)$ for the Dimensionsgrad:

$$\begin{aligned}(3)\quad &(\forall x)(\forall y)(\exists u)\\ &\bigl[\bigl((x\leqslant a)\wedge(y\leqslant a)\wedge(x\sqcap y=\mathbf{0})\bigr)\to\bigl(\operatorname{cut}(u,x,y,a)\wedge\Delta_{n-1}(u)\bigr)\bigr],\end{aligned}$$

and, as above, $\Delta_{-1}(a)$ denotes $a=\mathbf{0}$.

As with the large inductive dimension one has $\operatorname{Dg}X\leqslant n$ if and only if 2^X satisfies $\Delta_n(\mathbf{1})$. The same example as above shows that it is possible to have $\operatorname{Dg}X=1$ while some lattice base for the closed does not satisfy $\Delta_n(\mathbf{1})$ for any n. It is however also possible that some lattice base for the closed sets of a space X satisfies $\Delta_0(\mathbf{1})$ while $\operatorname{Dg}X>0$. An example is provided by the unit interval and the lattice base generated by $\bigl\{[0,q]\cup\{q+2^{-n}:n\in\omega\}:q \text{ rational}\bigr\}\cup\bigl\{[p,1]\cup\{p-2^{-n}:n\in\omega\}:p \text{ irrational}\bigr\}$. This lattice base satisfies $\Delta_0(\mathbf{1})$ *vacuously*, as it has no non-trivial connected elements.

4. Elementarity

In this section we fix a compact Hausdorff space X and an elementary sublattice L of the lattice 2^X, with Wallmany representation wL.

4.1. $\dim wL = \dim X$**.** This is by and large well-known but to keep this note self-contained we indicate a proof. By the remarks in the previous section we know that $\dim wL$ is the minimum natural number n for which L satisfies δ_n. Therefore we have to show that L satisfies δ_n if and only 2^X satisfies δ_n. The straightforward part is sufficiency: if 2^X satisfies δ_n then so does L: every $n+2$-tuple $(x_1,\ldots,x_{n+2})$ from L determines, via δ_n, an equation that has a solution $(y_1,\ldots,y_{n+2})$ in 2^X and hence in L. The converse follows by contraposition: the negation of δ_n is in itself an equation with only $\mathbf{0}$ and $\mathbf{1}$ as its parameters and unknowns $x_1, \ldots x_{n+2}$; if it has a solution in 2^X then it also has a solution in L.

4.2. $\operatorname{Ind} wL \leqslant \operatorname{Ind} X$**.** As above one deduces that L satisfies $I_n(\mathbf{1})$ if and only if 2^X satisfies $I_n(1)$: both $I_n(1)$ and its negation give rise to equations with parameters in L and solutions in 2^X, hence in L. In 3.2 we have seen that $\operatorname{Ind} wL\leqslant n$ whenever L satisfies $I_n(1)$; this suffices for $\operatorname{Ind} wL \leqslant \operatorname{Ind} X$.

4.3. $\operatorname{Dg} wL\leqslant\operatorname{Dg} X$**.** As above we find that L satisfies $\Delta_n(1)$ if and only if 2^X satisfies $\Delta_n(1)$. However, in 3.3 we saw that $\operatorname{Dg} wL \leqslant n$ does not follow automatically from the fact that L satisfies $\Delta_n(1)$. This shows that a bit more effort will have to go into the proof; in fact we shall prove the following proposition by induction on n.

Proposition 4.1. *Let X be a compact Hausdorff space with $\operatorname{Dg} X \leqslant n$ and L an elementary sublattice of 2^X. Then $\operatorname{Dg} wL \leqslant n$.*

Proof. In this proof an element A of L is on the one hand a closed subset of X and on the other hand a name for a basic closed set in wL; we write A_L to denote the latter set.

Let P and Q be closed and disjoint sets in wL. Because L is a lattice base for the closed sets of wL there are disjoint $A, B \in L$ with $P \subseteq A_L$ and $Q \subseteq B_L$.

Now in X there a cut C between A and B with $\operatorname{Dg} C \leqslant n-1$, by elementarity we can assume $C \in L$. Indeed, apparently there is in 2^X a solution to the equation $\operatorname{cut}(x, A, B, \mathbf{1}) \wedge \Delta_{n-1}(x)$, which has parameters in L, hence such a solution must exist in L.

We must show that the closed set C_L represented by C in wL is a cut between A_L and B_L (hence between P and Q) and that $\operatorname{Dg}(C_L) \leqslant n-1$.

The latter follows by induction because C_L is the Wallman representation of the lattice $\{x \in L : x \subseteq C\}$ and because this lattice is an elementary sublattice of $\{x \in 2^X : x \subseteq C\}$.

To prove the former assume K is a closed set in wL that meets A_L and B_L but not C_L. Take $H \in L$ with $K \subseteq H_L$ and $H \cap C = \emptyset$. Observe that H is not connected because it intersects both A and B but not C. One can therefore apply elementarity to the formula $\neg\operatorname{conn}(H)$ to find non-zero disjoint elements F and G of L with $H = F \cup G$. Then H_L is the disjoint union of F_L and G_L; this does not help in proving K disconnected however, as it is quite possible that $K \subseteq F_L$ or $K \subseteq G_L$. We shall have to choose F and G with extra care.

We use the fact that, in X, no component of H meets both A and B. Because the decomposition of H into its components is upper-semicontinuous [2, 6.2.21] it follows that we can find two disjoint closed sets F and G such that $F \cup G = H$,

$A \cap H \subseteq F$ and $B \cap H \subseteq G$. Again, elementarity dictates that there are such F and G in L.

Now work in wL: as $K \subseteq H_L$, we know that $\emptyset \neq K \cap A_L \subseteq H_L \cap A_L \subseteq F_L$ and $\emptyset \neq K \cap B_L \subseteq H_L \cap B_L \subseteq G_L$. But this implies K is not connected, as $K \subseteq H_L = F_L \cup G_L$ and $F_L \cap G_L = \emptyset$.

This shows, by contraposition, that C_L is indeed a cut between A_L and B_L. $\square$

4.4. **Proof of** $\dim X \leqslant \operatorname{Dg} X$**.** By the Löwenheim-Skolem theorem one can find a *countable* elementary sublattice M of 2^X. The Wallman representation wM of this lattice is compact and metrizable.

The theorem from [4] says that $\dim wM = \operatorname{Dg} wM$. Combined with the equality $\dim wM = \dim X$ and the inequality $\operatorname{Dg} wM \leqslant \operatorname{Dg} X$ this establishes $\dim X \leqslant \operatorname{Dg} X$.

5. Concluding remarks

As every partition between two closed sets is also a cut between these sets one gets the inequality $\operatorname{Dg} X \leqslant \operatorname{Ind} X$ for normal spaces without any real effort. A consequence of Fedorchuk's inequality is Vedenissof's inequality $\dim X \leqslant \operatorname{Ind} X$ for compact spaces [2, 7.2.8]. The Löwenheim-Skolem method can also be used to prove this directly: with the notation as in 4.4 one has $\dim X = \dim wM = \operatorname{Ind} wM \leqslant \operatorname{Ind} X$ (an application of the Čech-Stone compactification allows one to extend this to all normal spaces).

As remarked in the introduction the standard proof of $\dim X \leqslant \operatorname{Ind} X$ is fairly straightforward, whereas Fedorchuk's proof in [3] of $\dim X \leqslant \operatorname{Dg} X$ is longer and needs a closing-off argument to find a good cut. This difference is also apparent in the proofs in the present paper: in both cases the first step was to produce a (countable) lattice M that satisfies $I_n(\mathbf{1})$ or $\Delta_n(\mathbf{1})$ respectively. The second step was to deduce that $\operatorname{Ind} wM \leqslant n$ or $\operatorname{Dg} wM \leqslant n$ respectively. In either case the formula produced a candidate partition or cut; the problem then was to show that this set was indeed a partition or cut in the space wM. This is easy in the case of a partition: once the closed sets F and G are found we are done. In the case of a cut we only know that our set meets the connected elements of the base M that meet A and B; we need to know that the same holds for all continua in wM. This is where elementarity was used once more: it ensured that M already contained enough connected elements for the proof to go through. The reason for the perceived unwieldiness of Dg therefore seems to stem from the hidden universal quantifiers in the formula $\operatorname{cut}(u, x, y, a)$

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