c-20 Function Spaces

Given topological spaces X and Y we use $C(X, Y)$ to denote the set of all ***continuous maps*** from X to Y. One can define various topologies on $C(X, Y)$ – one can always take the ***discrete*** and ***indiscrete*** topologies – but for a workable theory of function spaces there should be some relation with the given topologies on X and Y.

1. Desirable topologies

One approach starts with the set Y^X of *all* maps from X to Y and the observation that there is a natural bijection between $Y^{X\times Z}$ and $(Y^X)^Z$: to $f : X \times Z \to Y$ associate the map $\Lambda(f) : Z \to Y^X$, defined by

$$\Lambda(f)(z)(x) = f(x, z).$$

The best one can hope for is that Λ induces a bijection between $C(X \times Z, Y)$ and $C(Z, C(X, Y))$. This problem splits naturally into two subproblems and hence into two definitions.

A topology on $C(X, Y)$ is said to be **proper** [E] or **splitting** [2] if for every space Z the map Λ maps $C(X \times Z, Y)$ into $C(Z, C(X, Y))$. If, conversely, Λ^{-1} always maps $C(Z, C(X, Y))$ into $C(X \times Z, Y)$ then the topology is called **admissible** [E], **jointly continuous** [Ke, N] or **conjoining** [2]. A topology that has both properties is called **acceptable** [E].

Every topology ***weaker*** than a proper topology is again proper and every topology ***stronger*** than an admissible topology is again admissible. Every proper topology is weaker than every admissible topology, hence there can be only one acceptable topology.

Also, a topology on $C(X, Y)$ is admissible iff it makes the evaluation map $(f, x) \mapsto f(x)$ from $C(X, Y) \times X$ to Y continuous. Furthermore, the join of all proper topologies is proper, hence there is always a largest proper topology on $C(X, Y)$.

2. The topology of pointwise convergence

The **topology of pointwise convergence** τ_p is simply the ***subspace topology*** that $C(X, Y)$ receives from the ***product topology*** on Y^X. The name comes from the fact that a ***net*** $(f_\alpha)_{\alpha\in D}$ converges with respect to τ_p iff it converges pointwise. In keeping with the other articles on this topology, we write $C_p(X, Y)$ to indicate that we use τ_p. This topology is proper but in general not admissible: $g \in C(Z, C_p(X, Y))$ means that $\Lambda^{-1}(g)$ is ***separately continuous***, whereas $\Lambda^{-1}(g) \in C(X \times Z, Y)$ means that it is ***jointly continuous***.

3. The topology of uniform convergence

If Y is a ***metric space*** or, more generally, a ***uniform space*** then one can define on $C(X, Y)$ the **topology of uniform convergence** τ_u, which can be defined by stipulating that a net of functions converges with respect to τ_u iff it **converges uniformly**, i.e., in the metric case $f_\alpha \to f$ iff for every $\varepsilon > 0$ there is an α such that $d(f_\beta(x), f(x)) < \varepsilon$ whenever $\beta \geqslant \alpha$ and $x \in X$. In the case of a uniform space this becomes: for every ***entourage*** U there is an α such that $(f_\beta(x), f(x)) \in U$ whenever $\beta \geqslant \alpha$ and $x \in X$. A ***local base*** at f is given by sets of the form $B(f, U) = \{g\colon (f(x), g(x)) \in U \text{ for all } x\}$, where U runs through the family of entourages.

This topology is admissible but in general not proper: Consider $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, y) = xy$; the map $\Lambda(f) : \mathbb{R} \to C_u(\mathbb{R}, \mathbb{R})$ is not continuous.

4. The compact-open topology

The previous topologies are, in general, not acceptable. This is, to some extent, to be expected as neither depends on the topology of X (though the *set* $C(X, Y)$ does depend on X). The **compact-open topology** τ_c does depend on the topologies of both X and Y. It is defined by specifying a ***subbase***: we take all sets of the form

$$[K, O] = \{f\colon f[K] \subseteq O\}$$

where K runs through the ***compact*** subsets of X and O through the open sets of Y. We write $C_c(X, Y)$ to indicate that $C(X, Y)$ carries the compact-open topology.

The compact-open topology is always proper; it is admissible and hence acceptable if X is ***locally compact*** and ***Hausdorff***. Local compactness is the crucial property as one can show that $C(\mathbb{Q}, \mathbb{R})$ carries no acceptable topology and, stronger, if X is ***completely regular*** and $C(X, \mathbb{R})$ carries an acceptable topology then X is locally compact.

5. Properties and relations

Any property that is ***productive*** and ***hereditary*** is transferred from Y to $C_p(X, Y)$, this includes the separation axioms up to and including complete regularity. The same holds for $C_c(X, Y)$; this follows because $\tau_p \subseteq \tau_c$ (for properties below ***regularity***) but requires extra proof for the other properties.

If X is discrete then $C(X, Y) = Y^X$, and τ_p and τ_c coincide with the product topology; this shows that normality and stronger properties in general do not carry over.

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In case Y is a metric (or uniform) space then τ_c may also be described as the **topology of uniform convergence on compact subsets**: for every f the sets of the form

$$B(f, K, \varepsilon) = \{g\colon (\forall x \in K)\big(d\big(f(x), g(x)\big) < \varepsilon\big)\}$$

where K is compact in X and $\varepsilon > 0$ form a local base for τ_c at f (with obvious modifications in case of a uniform space). In particular, if X is compact the compact-open topology and the topology of uniform convergence coincide.

One obtains a local base also if K runs through a ***cofinal*** family of compact sets. This means that, if Y is a metric space, τ_c is a ***first-countable*** topology if X is **hemicompact**, which means that there is a countable cofinal family of compact sets; in fact τ_c is even metrizable. In general the ***character*** of τ_c is the product of the cofinality of the family of compact sets of X and the ***weight*** of the uniform space Y.

The compact-open topology makes $C(X, \mathbb{R})$ into a ***topological group***; this group is ***complete*** in its natural ***uniformity*** iff the space X is a k_R***-space***.

6. Compactness in function spaces

Compactness is a very useful property and it is desirable to have characterizations of compactness for subsets of $C(X, Y)$. The classical **Arzèla–Ascoli Theorem** states that a subset F of $C([0,1], \mathbb{R})$ has a compact closure with respect to τ_u if it is equicontinuous and bounded. This theorem admits various generalizations.

If Y is a uniform space we define a subset F of $C(X, Y)$ to be **equicontinuous** if for every entourage U and every $x \in X$ there is one neighbourhood O of x such that $(f(x), f(y)) \in U$ for all $f \in F$ and $y \in O$. Then, if X is a k***-space*** a closed subset F of $C_c(X, Y)$ is compact iff it is equicontinuous and for each x the set $\{f(x)\colon f \in F\}$ has compact closure.

A similar result holds for regular Y; one has to replace equicontinuity by even continuity, where a subset F of $C_c(X, Y)$ is **evenly continuous** if for every $x \in X$, every $y \in Y$ and every neighbourhood V of y there are neighbourhoods U and W of x and y respectively such that for every f the implication $f(x) \in W \Longrightarrow f[U] \subseteq W$ holds.

As noted above, if Y is metrizable and X is hemicompact then $C_c(X, Y)$ is metrizable. To define a metric one takes a bounded metric d on Y and a cofinal family $\{K_n\colon n \in \mathbb{N}\}$ of compact sets in X. For each n the formula $d_n(f, g) = \sup\{d(f(x), g(x))\colon x \in K_n\}$ defines a ***pseudometric*** in $C(X, Y)$. The sum $\rho = \sum_n 2^{-n} d_n$ is a metric on $C(X, Y)$ that induces the compact-open topology. This shows that the metric topology on the set $H(U)$ of holomorphic functions on a domain U that one encounters in complex function theory is the compact-open topology. Thus the Arzèla–Ascoli theorem provides an inroad to Montel's theorem on normal families of analytic functions.

7. The Stone–Weierstrass Theorem

The familiar **Stone–Weierstrass Theorem** states that for a compact space X every subring of $C(X, \mathbb{R})$ that contains all constant functions and separates the points of X is ***dense*** with respect to the topology of uniform convergence.

This theorem characterizes compactness among the completely regular spaces: if X is not compact then take a point x in X and a point z in $\beta X \setminus X$; the subring $\{f\colon \beta f(z) = f(x)\}$ of $C(X, \mathbb{R})$ is not dense, though it satisfies the conditions in the Stone–Weierstrass Theorem.

There is also a version of this theorem for general spaces: any subring as in the Stone Weierstrass theorem is always dense in $C_c(X, \mathbb{R})$.

8. More topologies

There are many more ways to introduce topologies in $C(X, Y)$, we mention two.

Variations on the compact-open topology
Given spaces X and Y one can first specify families $\mathcal{K}$ and $\mathcal{O}$ of subsets of X and Y respectively and then simply decree that

$$\{[K, O]\colon K \in \mathcal{K},\ O \in \mathcal{O}\}$$

be a subbase for a topology on $C(X, Y)$. The compact-open topology arises when $\mathcal{K}$ is the family of compact subsets of X and $\mathcal{O}$ the topology of Y; changing $\mathcal{K}$ to the finite subsets of X then yields the topology of pointwise convergence.

Hyperspace topologies
If Y is Hausdorff then the graph of every continuous map from X to Y is a closed subset of $X \times Y$, so $C(X, Y)$ is in a natural way a subset of the ***hyperspace*** $2^{X \times Y}$. Thus, any hyperspace topology immediately gives rise to a function space topology.

The volume [1] contains systematic studies of the somewhat bewildering array of topologies obtained in this way.

References

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[3] R.A. McCoy and I. Ntantu, *Topological Properties of Spaces of Continuous Functions*, Lecture Notes in Math., Vol. 1315, Springer, Berlin (1988).

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