d-16 Remainders

In full generality a **remainder** of a space X is a space of the form $Y \setminus X$, where Y is an **extension** of X, i.e., a space containing (a homeomorphic copy of) X. Normally the extension comes from a class of nice spaces, e.g., compact Hausdorff or completely metrizable and we assume that X is dense in the extension.

1. Remainders in compactifications

In this section we consider remainders in ***compactifications***, so we assume X is ***completely regular*** and we consider compact Hausdorff extensions. In this case the remainders of X are the spaces $\gamma X \setminus X$ where γX runs through the family of compactifications of X. Every remainder is the image of $X^* = \beta X \setminus X$ (the **Čech–Stone remainder**) under a ***perfect map*** – to wit the restriction of the natural map from βX onto γX. Unfortunately the converse is not true in general. It is true if X is ***locally compact***, this is generally known as **Magill's theorem** (see [E, 3.5.13]) but a general characterization is still lacking.

Fundamental questions that have driven the research in this area through the years are: what are the properties of X^*, what are the remainders of specific X, what spaces have 'nice' remainders?

The first question is too ambitious as "every space is a remainder"; indeed, given Y put

$$X = \big((\omega_1 + 1) \times \beta Y\big) \setminus \big(\{\omega_1\} \times Y\big),$$

then $Y = X^*$. Therefore one should modify it with "as a function of X". Observe that the space X just constructed is ***pseudocompact***; this is why one looks for general results on X^* in classes that do not contain pseudocompact spaces. A first such result is that if X is not pseudocompact then X^* contains a copy of $\mathbb{N}^*$, which tells us something about the size of X^* – at least $2^{\mathfrak{c}}$ – and its structure – at least as complicated as $\mathbb{N}^*$.

This also implies that, for non-pseudocompact X at least, recognizing the remainders of X is at least as hard as recognizing those of $\mathbb{N}$. In the article on $\beta\mathbb{N}$ and $\beta\mathbb{R}$ in this volume Parovičenko's theorem is quoted, which says that under CH the continuous images of $\mathbb{N}^*$, and hence the remainders of $\mathbb{N}$, are precisely the compact spaces of weight $\mathfrak{c} = \aleph_1$. In ZFC alone one has to work harder. Thus far the following have been shown to be remainders of $\mathbb{N}$: all compact spaces of weight $\aleph_1$, all separable compact spaces, all perfectly normal compact spaces and certain products of spaces from these classes. If $f : X \to Y$ is a perfect surjection then $\beta f^{-1}[Y^*] = X^*$ (and conversely); in this case every remainder of Y is also a remainder of X. Therefore, if X is the sum of countably many compact spaces then every remainder of $\mathbb{N}$ is a remainder of X. This fact may also be used to see that, for example, $[0,\infty)$ and $\mathbb{R}^n$ (where $n > 2$) share the same remainders: the map $x \mapsto \|x\|$ is perfect from $\mathbb{R}^n$ to $[0,\infty)$ and there is also a perfect 'space-filling curve' from $[0,\infty)$ onto $\mathbb{R}^n$. The remainder $\mathbb{R}^*$ is not connected so the remainders of $\mathbb{R}$ form a larger family than those of $[0,\infty)$.

For locally compact spaces ***Wallman–Shanin compactifications*** offer a way to recognizing shared remainders: if $\mathcal{B}$ is a ***Wallman base*** for the locally compact space X and $w(X,\mathcal{B})$ is the associated compactification then the remainder $w(X,\mathcal{B}) \setminus X$ is the ***Wallman representation*** of the quotient lattice of $\mathcal{B}$ by the following equivalence relation: "every element of $\mathcal{B}$ contained in $A \triangle B$ is compact".

There are various other results on shared remainders but no general pattern has emerged. By way of example we mention that $\mathbb{Q}$, $\mathbb{P}$ and $\mathbb{S}$ share no remainders: all remainders of $\mathbb{Q}$ are ***topologically complete***, all remainders of $\mathbb{P}$ are of ***first category*** and all remainders of $\mathbb{S}$ are ***Baire spaces*** but not complete.

Discrete spaces offer an intriguing problem. It is known that if there are distinct cardinals κ and λ with κ^* and λ^* homeomorphic then also ω^* and ω_1^* are homeomorphic. It is still an open question whether "ω^* and ω_1^* are homeomorphic" is consistent with ZFC.

A very useful general result on remainders is that whenever X is ***σ-compact*** and ***locally compact*** (but not compact) then the remainder X^* is a compact ***F-space*** in which non-empty ***G_δ-sets*** have non-empty interior. This immediately implies that such X^* are not ***homogeneous***; in fact if X is not pseudocompact then X^* is never homogeneous.

2. Nice remainders

Another line of investigation is to start with a class of 'nice' spaces and to see which spaces have compactifications with a remainder from that class.

Locally compact spaces, and only those, are the spaces that have finite remainders. Demanding that *all* remainders of a space be finite brings us back to pseudocompactness again. The statement $|X^*| = n$ translates into: of every $n+1$ mutually ***completely separated*** closed sets at least one must be compact. If $|X^*| = 1$ then X is called **almost compact**.

One of the first results on nice remainders is Zippin's theorem that a separable metric space has a compactification with a countable remainder (in short: it is a **CCR space**) iff it is both ***Čech-complete*** and ***rim-compact***. Freudenthal showed that a separable metric space has a compactification with a ***zero-dimensional*** remainder (in short: it is a **0-space**)

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iff it is rim-compact. Both proofs provide implications in the class of completely regular spaces: all CCR spaces are Čech-complete and rim-compact, and all rim-compact spaces are 0-spaces (although rim-compact spaces needs not have ***strongly zero-dimensional*** remainders).

These implications are, in general, not reversible: there are 0-spaces that are not rim-compact and the product of the space of irrationals with an uncountable discrete space is completely metrizable and zero-dimensional but not a CCR space. There have been attempts to characterize CCR spaces and 0-spaces in terms of the set $R(X)$ of points without compact neighbourhoods. Adding separability of $R(X)$ to the necessary conditions of Čech-completeness and rim-compactness yields a characterization of CCR spaces among metrizable spaces. For general spaces adding the assumption that $R(X)$ is the continuous image of a separable metric space (a ***cosmic space***) ensures that X is a CCR space. However, intrinsic properties alone of $R(X)$ will not provide the definitive answer as there are two Čech-complete, even zero-dimensional, spaces X and Y, with both $R(X)$ and $R(Y)$ discrete and uncountable, yet X is a CCR space but Y is not.

For 0-spaces some conditions on $R(X)$ suffice to ensure rim-compactness: local compactness plus zero-dimensionality or ***scatteredness***. On the other hand there are spaces X and Y with $R(X)$ and $R(Y)$ homeomorphic to the space of irrationals of which X has a ***totally disconnected*** remainder but is not a 0-space and Y is a 0-space that is not rim-compact. Thus, again, a complete solution in terms of $R(X)$ seems unlikely.

Freudenthal's proof provides, for rim-compact spaces, a canonical construction of a compactification with zero-dimensional remainder, the ***Freudenthal compactification***. The Freudenthal compactification is the unique **perfect compactification** with zero-dimensional remainder and it is also the minimum perfect compactification. A compactification αX is perfect if $\operatorname{cl}\operatorname{Fr} O = \operatorname{Fr}\operatorname{Ex} O$ for all open subsets of X. This is equivalent to saying that the natural map from βX onto αX is ***monotone***. In general X has a minimum perfect compactification μX iff it has some compactification with a ***punctiform*** remainder and in that case μX is also the maximum compactification with punctiform remainder.

A particularly interesting family is formed by what are commonly referred to as Ψ**-space**s. These are built by taking a set X and an ***almost disjoint family*** $\mathcal{A}$ consisting of countably infinite sets. The space $\Psi(X, \mathcal{A})$ has $X \cup \mathcal{A}$ as its underlying set, the points of X are isolated and for $A \in \mathcal{A}$ a typical basic neighbourhood is of the form $\{A\} \cup A \setminus F$ for some finite set F. Thus, the set A becomes a converging sequence with limit A. We concentrate on the case where $X = \omega$ and $\mathcal{A}$ is a maximal almost disjoint (MAD) family; we abbreviate $\Psi(\omega, \mathcal{A})$ by $\Psi(\mathcal{A})$. The space $\Psi(\mathcal{A})$ is pseudocompact and locally compact. Every compact metric space is the Čech–Stone remainder of some $\Psi(\mathcal{A})$. This implies that there are $\Psi(\mathcal{A})$ with infinite ***covering dimension*** as well as almost compact $\Psi(\mathcal{A})$. The Continuum Hypothesis implies that the families of all remainders of $\mathbb{N}$ and of all $\Psi(\mathcal{A})^*$ coincide. It is consistent, however, that there is an $\mathcal{A}$ for which some continuous image of $\Psi(\mathcal{A})^*$ is not of the form $\Psi(\mathcal{B})^*$ for any $\mathcal{B}$.

3. Dimension

An interesting line of research was opened by de Groot when he asked for an internal characterization of the minimum dimension of remainders in compactifications. This number is called the **compactness defect** or **compactness deficiency** of the space: $\operatorname{def} X = \min\{\dim Y \setminus X\colon Y \text{ is a compactification of } X\}$. The problem is usually studied in the class of separable metric spaces, so it does not matter what dimension function is used.

The **compactness degree** is defined just like the ***small inductive dimension***: $\operatorname{cmp} X \leqslant n + 1$ if for every point x and every open $O \ni x$ there is an open U with $x \in U \subseteq \operatorname{cl} U \subseteq O$ and $\operatorname{cmp} \operatorname{Fr} U \leqslant n$, the starting point however is not the empty set but the class of compact spaces: $\operatorname{cmp} X = -1$ means that X is compact. De Groot proved $\operatorname{cmp} X = \operatorname{def} X$ for values up to 0 and $\operatorname{cmp} X \leqslant \operatorname{def} X$ for all (separable metric) spaces. In 1982 R. Pol disproved de Groot's conjecture $\operatorname{cmp} = \operatorname{def}$ with an example of a space X with $\operatorname{cmp} X = 1$ and $\operatorname{def} X = 2$. Since then more examples appeared to show that the gap may be arbitarily large and that $\operatorname{Cmp} X$ (defined like the ***large inductive dimension***) does not characterize $\operatorname{def} X$ either. In 1988 T. Kimura showed that an invariant due to Skljarenko does characterize $\operatorname{def} X$. One has $\operatorname{def} X \leqslant n$ iff X has a base $\mathcal{B}$ such that whenever $\mathcal{B}'$ is an $n + 1$-element subfamily of $\mathcal{B}$ the intersection $\bigcap_{B \in \mathcal{B}'} \operatorname{Fr} B$ is compact.

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