d-23 Special Constructions

General Topology is rife with examples; the article on *Special spaces* contains a selection of spaces that 'every young topologist should know'. Many of the articles in this volume describe or refer to further examples, some specific to a problem and others of general use. The present article describes some techniques for constructing spaces that have become part of the topologist's standard toolkit.

1. Spaces of maximal systems

A versatile and oft-used method for constructing topological spaces is by means of maximal systems. Various instances of this method can be found in other articles in the volume, we mention the ***Stone space*** of a ***Boolean algebra*** and the ***Wallman representation*** of a distributive lattice. Both have as points the maximal ***filter***s on the structure. For an element a of the structure let a^+ denote the set of all maximal filters that contain a, i.e., $a^+ = \{x\colon a \in x\}$; the family of all a^+ is used as a ***subbase*** for the closed sets of the space, in the cases of lattices (and hence Boolean algebras) one even obtains a ***base***.

The Stone space

The ***Stone duality*** between Boolean algebras and ***compact zero-dimensional*** spaces provides a rich source of objects on either side, with neither side the clear winner. The ***measure algebra*** gives us a compact ccc non-separable space courtesy of this duality.

Various compactifications of $\mathbb{N}$ have been constructed by constructing suitable subalgebras of $\mathcal{P}(\mathbb{N})$, see [KV, Chapter 11] for Bell's compactification $\gamma\,\mathbb{N}$ of $\mathbb{N}$ with $\gamma\,\mathbb{N} \setminus \mathbb{N}$ ccc non-separable and [14] for more examples.

One way of defining the **absolute** (or **Gleason space**) of a topological space X is by way of the Stone space of the algebra of ***regular open*** sets. The absolute EX is the set of converging ultrafilters and the natural map $\pi_X : EX \to X$ simply assigns the limit to each ultrafilter. See [12] for a thorough treatment of this subject.

Cliques in graphs and hypergraphs

The Stone space of a Boolean algebra B can in a natural way be identified with a subspace of the ***Cantor cube*** $\{0,1\}^B$; indeed, the ultrafilter x corresponds to the associated homomorphism $\varphi_x : B \to \{0,1\}$, defined by $\varphi_x(a) = 1$ iff $a \in x$. The subspace topology corresponds to the Stone space topology.

This idea can be used for other structures as well; we consider one very successful instance, that of cliques in graphs and hypergraphs.

A **graph** consists of a set V of points and a subset E of $[V]^2$, called the edges. A **clique** is a subset C of V such that $[C]^2 \subseteq E$. One immediately obtains the space of maximal cliques, as the points x of 2^V for which $x^{\leftarrow}(1)$ is a maximal clique. This space is zero-dimensional but not necessarily compact; its properties are, however, intimately connected with the partial order of finite cliques.

In [10] one finds a construction, using CH, of a two complementary graphs E_1 and E_2 on ω_1 such that the spaces of maximal cliques satisfy the ccc, but their product does not. The ccc translates into the following statement: whenever $\mathcal{F}$ is an uncountable family of finite cliques there are $F, G \in \mathcal{F}$ such that $F \cup G$ is a clique as well. Because the graphs are complementary the basic clopen sets $\{(x,y)\colon x(\alpha) = y(\alpha) = 1\}$ in the product are disjoint.

The space of *all* cliques (plus the empty set) is compact [1] and it has been used to produce a first-countable compact space that is not a continuous image of $\mathbb{N}^*$ (the graph is a Cohen-generic subset of ω_2).

Whether a set is a clique depends on its finite subsets. A family of sets with this property has been called an **adequate family**: a family $\mathcal{A}$ of subsets of a set V is adequate if every singleton belongs to it, it is closed under taking subsets and $A \in \mathcal{A}$ iff every finite subset of A belongs to $\mathcal{A}$. Such families have been used to construct various interesting compact spaces, see [13] and [KV, Chapter 23] for applications to the theory of Banach spaces.

Maximal threads in complexes

It is possible to put a compact topology on the family of maximal elements of an adequate family; this was done in [16] and is more in the spirit of the Stone representation: given a set V, an **abstract complex** is a family K of finite subsets of V such that $t \in K$ and $s \subseteq t$ always implies $s \in K$. A **thread** in the complex is a subset W of V with $[V]^{<\aleph_0} \subseteq K$ (so the set of threads is an adequate family). The set of all maximal threads can be topologized by using the family $\mathcal{S} = \{v^+\colon v \in V\}$ as a subbase for the closed sets, where v^+ denotes the set of maximal threads that contain v. The resulting space is compact and T_1 (not necessarily Hausdorff) and every compact ***T_1-space*** can be obtained in this way (see the article on *Wallman–Shanin compactification* in this volume).

In case V is a graph and K is the family of all finite cliques this yields a compact T_1 topology on the set of all maximal cliques; in fact this topology is ***supercompact***: the natural subbase $\mathcal{S}$ is a ***binary subbase***, for a family $\{v^+\colon v \in W\}$ is linked iff W is a clique. See *Topological Characterizations of Spaces* for an application.

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The Hull–Kernel topology
Given a ring R let M_R denote the set of its maximal ideals. The **hull-kernel topology** is defined by specifying a ***closure operator***, thus: if I is a set of maximal ideals then its closure is defined to be $\{m \in M\colon \bigcap I \subseteq m\}$, the 'hull' of the 'kernel'.

When applied to the ring $C(X)$ of real-valued continuous functions on the ***completely regular*** space X one obtains the ***Čech–Stone compactification*** of X.

2. Recursive constructions

Lots of examples have been constructed by recursion. The basic idea is to construct a topology on a well-ordered set X by specifying neighbourhoods at each point, using the neighbourhoods of the points that come before in the well-order. We mention some well-known examples.

Ostaszewski's space
This is a ***perfectly normal countably compact*** noncompact space. The underlying set is ω_1 with its natural order; the main ingredients in the construction are an enumeration of $[\omega_1]^{\aleph_0} = \{A_\alpha\colon \alpha \text{ a limit}\}$ (with $A_\alpha \subseteq \alpha$) and a ***♣-sequence*** $\{B_\alpha\colon \alpha \text{ a limit}\}$ – this construction is under the assumption of the $\Diamond$-principle. We define a topology τ_α on α, such that (β, τ_β) is an open subspace of (α, τ_α) whenever $\beta < \alpha$.

When α is a limit we first let τ_α be the ***direct limit*** topology of the topologies τ_β with $\beta < \alpha$, i.e., $O \in \tau_\alpha$ iff $O \cap \beta \in \tau_\beta$ for all $\beta < \alpha$. Then we consider A_α and B_α; the latter set is closed and discrete in α. If A_α has an accumulation point in α we ignore it; otherwise we enumerate $A_\alpha \cup B_\alpha$ (in a one-to-one fashion) as $\{x_n\colon n \in \mathbb{N}\}$. Find a disjoint clopen cover $\{C_n\colon n \in \mathbb{N}\}$ of α such that $x_n \in C_n$ for all n. Finally then write $\mathbb{N}$ as a disjoint union $\bigcup_i D_i$ of infinite sets such that each D_i has infinitely many n with $x_n \in A_\alpha$ and with $x_n \in B_\alpha$. Finally then a local base at $\alpha + i$ consists of the sets $\{\alpha + i\} \cup \bigcup\{C_n\colon n \in D_i,\ n \geqslant m\}$ $(m \in \mathbb{N})$.

One checks that $\operatorname{cl} B_\alpha = B_\alpha \cup [\alpha, \omega_1)$ for all α, that every countably infinite subset has an accumulation point and that every set α is open. This gives us the desired properties of the space. In the course of the construction one should verify that (α, τ_α) is always metrizable and, if one starts in the right way, locally compact; this will enable one to keep going. Observe that the ♣-principle also implies that the resulting space is hereditarily separable and hence an S-space.

Kunen's line
This is an S-space topology on the real line. Its construction is like the previous one, but using a bit of the structure of $\mathbb{R}$. The main ingredient is an enumeration $\{A_\alpha\colon \alpha < \omega_1\}$ of the family of countably infinite subsets of $\mathbb{R}$ and an enumeration $\{x_\alpha\colon \alpha < \omega_1\}$ of $\mathbb{R}$ itself – this construction works under CH. With the aid of the metric structure one can now recursively define local bases at x_α so that the following holds in the end: if $\beta < \alpha$, $A_\beta \subseteq \{x_\gamma : \gamma < \alpha\}$ and $x_\alpha \in \operatorname{cl}_{\mathbb{R}} A_\beta$ then $x_\alpha \in \operatorname{cl} A_\beta$.

This makes the resulting space hereditarily separable: if $A \subseteq \mathbb{R}$ then $A \subseteq \operatorname{cl}_{\mathbb{R}} A_\alpha$ for some α and then $A_\alpha \cup (A \cap \{x_\beta\colon \beta < \alpha\})$ is dense in A.

Van Douwen's line
This is a topology on the real line that is constructed much like in the previous example but in ZFC only. The main ingredient now is a listing $\{\langle K_{\alpha,n}\colon n \in \mathbb{N}\rangle\colon \alpha < \mathfrak{c}\}$ of all sequences of countable subsets of $\mathbb{R}$ with $\bigcap_n \operatorname{cl}_{\mathbb{R}} K_{\alpha,n}$ uncountable. As above, the construction is set up to retain this property and to make the topology locally compact and locally countable. The resulting space is ***normal***, ***countably paracompact***, ***separable*** but not ***paracompact*** and not hereditarily normal. This idea has proved very versatile; van Douwen's [5] contains many more applications and offers a good introduction to this method.

3. Resolutions

A totally different type of construction is the resolution, which is a way of replacing each point in a space by a copy of some other space, possibly a different one for each point. This is done as follows. We are given a space X, and for each $x \in X$ a space Y_x and a continuous map $f_x : X \setminus \{x\} \to Y_x$. The **resolution** of X (at each x, into Y_x, by f_x) is the set $R(X, Y_x, f_x) = \bigcup_{x \in X} \{x\} \times Y_x$.

To define the topology we define for each pair (U, V), where U is open in X and V is open in Y_x for some $x \in U$, the set

$$U \otimes V = \{x\} \times V \cup \bigcup\bigl\{\{x'\} \times Y_{x'}\colon x' \in U \cap f_x^{-1}[V]\bigr\}.$$

The family of all such sets is a base for a topology on $R(X, Y_x, f_x)$, the **resolution topology**; we usually suppress mention of the spaces Y_x and the maps f_x and write $R(X)$ for the resolution. Generally the space X is assumed to be ***completely regular*** and the spaces Y_x are assumed to be compact.

The natural map $\pi : R(X) \to X$ (we shall call this a resolution too) is ***continuous*** and ***closed*** (if each Y_x is compact), and for each x the map $y \mapsto (x, y)$ is an embedding of Y_x into $R(X)$, so $R(X)$ is indeed obtained by replacing each x with Y_x. When all spaces involved are compact Hausdorff then so is the resolution (and conversely). The familiar $\sin\frac{1}{x}$-curve is a resolution: one takes $X = [0, 1]$, $Y_0 = [-1, 1]$ and $f_0(y) = \sin\frac{1}{y}$, and for $x > 0$ simply $Y_x = \{x\}$ and $f_x(y) = x$. The resolution process enables one to do this at all points of $[0, 1]$ at once: just take $Y_x = [-1, 1]$ and $f_x(y) = \sin\frac{1}{y-x}$ for all x. The resulting space is a first-countable ***chainable continuum***, whose ***small inductive dimension*** is two.

If each Y_x is compact, then the resolution map $\pi : R(X) \to X$ is a **fully closed map** (also called a **strongly closed map**); a map $f : X \to Y$ is fully closed if for every $y \in Y$ and every finite open cover $\mathcal{U}$ of $f^{-1}(y)$ the set $\{y\} \cup \bigcup\{f^{\#}[U]\colon U \in$

$\mathcal{U}\}$ is open, where $f^{\#}[U]$ denotes the **small image**, i.e., the set $\{z: f^{-1}(z) \subseteq U\} = Y \setminus f[X \setminus U]$.

A fully closed surjection $f : X \to Y$ between compact Hausdorff spaces is almost a resolution. The added requirement comes from considering, for every $y \in Y$ the decomposition $\{\{x\}: f(x) = y\} \cup \{f^{-1}(z): z \neq y\}$ of X and the quotient space Y^y (this is Y with just y replaced by $f^{-1}(y)$). If f is fully closed *and* if for every y the fibre $f^{-1}(y)$ is a retract of Y^y then f is a resolution map. This operation may also be used to characterize fully closed maps themselves: a perfect map $f : X \to Y$ with regular domain is fully closed iff every Y^y is regular.

Fully closed maps almost preserve ***covering dimension*** between domain and range. If $f : X \to Y$ is a fully closed map between normal spaces then

$$\dim Y \leqslant \dim X + 1$$

and

$$\dim X \leqslant \max\{\dim Y, \dim f\},$$

where

$$\dim f = \sup\{\dim f^{-1}(y): y \in Y\}.$$

The inductive dimensions of resolutions can be made high by using so-called ring maps: a surjective map $f : X \to Y$ is a **ring map** at $y \in Y$ if for every $x \in f^{-1}(y)$ and neighbourhoods O_x of x and O_y of y the set $O_y \cap f^{\#}[O_x]$ contains a partition between y and $Y \setminus O_y$; we say f is a ring map if it is a ring map at every point of Y. The resolution map $\pi : R(X) \to X$ is a ring map at $x \in X$ iff for every $y \in Y_x$ and neighbourhoods O_y of y and O_x of x the intersection $O_x \cap f_x^{-1}[O_y]$ contains a partition between x and $X \setminus O_x$.

The $\sin \frac{1}{x}$ resolutions described above are ring resolutions. In raising inductive dimensions the following result is often used. If $f : X \to Y$ is a monotone ring map of the compact space onto an n-dimensional ***Cantor manifold*** Y, where $n \geqslant 2$, then every partition of X contains some fibre of f. Thus resolving each point of $[0, 1]^n$ into $[0, 1]^n$ by means of ring maps yields an n-dimensional first-countable compact space without $(n-1)$-dimensional partitions. To do so fix a dense subset $\{d_n: n \in \mathbb{N}\}$ of $[0, 1]^n$ and for each x a local base $\{U_n^x: n \in \mathbb{N}\}$ with $\operatorname{cl} U_{n+1}^x \subseteq U_n^x$; by ***Urysohn's Lemma*** we can find $f_x : [0, 1]^n \setminus \{x\} \to [0, 1]^n$ such that $f_x[U_n^x] = \{d_n\}$. This can be iterated to produce an inverse sequence of spaces whose limit is first-countable and n-dimensional but whose closed subsets are either n- or zero-dimensional.

One can extend such constructions to inverse sequences of length ω_1; using the ***Diamond Principle*** one can then construct compact S-spaces of cardinality $2^{\mathfrak{c}}$, higher-dimensional versions of Ostaszewski's space and perfectly normal n-manifolds with $n < \dim < \mathrm{Ind}$.

It is clear from the definition that for each x the map f_x need only be defined on a neighbourhood of x. The resolution process can be generalized to replaced subsets by products. We are given a space X, a family $\{O_\alpha\}_\alpha$ of open subsets of X and for each α a subset G_α closed in O_α and a map $f_\alpha : O_\alpha \setminus G_\alpha \to Y_\alpha$. For each α let $X_\alpha = X \setminus G_\alpha \cup G_\alpha \times Y_\alpha$, topologized by using all sets of the form $U \setminus G_\alpha \cup (U \cap G_\alpha) \times Y_\alpha$ (U open in X) and $U \cap f_\alpha^{-1}[V] \cup (U \cap G_\alpha) \times V$ (U open in X and V open in Y_α) as a base for the open sets. The obvious map $\pi_\alpha : X_\alpha \to X$ is continuous; the resolution of X by all the maps f_α is the subspace of the product $\prod_\alpha X_\alpha$ consisting of those points x for which $\pi_\alpha(x_\alpha) = \pi_\beta(x_\beta)$ for all α and β. This type of resolution was used to construct a compactification that is not a Wallman–Shanin compactification.

Resolutions were defined by Fedorchuk in [7]; a comprehensive introduction with many examples is given in [HvM, Chapter 20]. Some more applications are described in the references. The generalized resolution was introduced by Ul'janov in [15].

4. Elementary substructures

Strictly speaking this is not a method of constructing examples but rather a general stratagem that helps one to avoid laborious inductive proofs and recursive constructions. An **elementary substructure** (or **elementary submodel**) of the universe V is a set M with the following property: if $n \in \mathbb{N}$, $(a_1, \ldots, a_n) \in M^n$ and φ is a set-theoretic formula such that there is some x in V for which $\varphi(x, a_1, \ldots, a_n)$ holds then there is a c in M for which $\varphi^M(c, a_1, \ldots, a_n)$ holds. Here φ^M denotes the formula φ but with every existential quantifier $\exists z$ replaced by $\exists z \in M$.

This notion takes some time to get used to, mainly because of its seemingly abstract nature. A helpful analogy is that of algebraic closure: every algebraic equation with algebraic numbers for parameters has its solutions, if any, in the set of algebraic numbers. If one interprets $\varphi(x, a_1, \ldots, a_n)$ as an equation with parameters from M, then elementarity says that if the equation has a solution then at least one of these solutions is in M and the fact that it is a solution can be checked within M. The Löwenheim–Skolem Theorem gives us a large supply of elementary substructures of V: for every set X there is an M with $X \subseteq M$ and $|M| \leqslant \aleph_0 \cdot |X|$. The proof of this theorem simply subsumes all inductions and recursions that we may wish to perform. Thus once the phrase "let M be an elementary substructure of the universe" is uttered we have already performed the construction we intended to perform. It takes a slightly different mindset to work this way but the two examples below may be contrasted with the standard proofs by recursion. Experience shows that once one gets into this 'elementarity mindset' one gains a powerful tool for discovering results and proofs that would otherwise stay out of reach. The main new aspect here is the interplay between the external and internal views of the substructure; this is something that is very hard to obtain in 'normal' inductive and recursive situations.

Dow's articles [6] and [HvM, Chapter 4] are a good places to start acquiring the mindset needed to work with elementary substructures of the universe.

Continuous functions on ω_1

The well-known theorem that any continuous function $f:\omega_1 \to \mathbb{R}$ is constant on a tail can easily be proved by elementarity. Simply take a *countable* elementary substructure of the universe with $f \in M$. The sets ω, ω_1 and $\mathbb{R}$ automatically belong to M because they are unique solutions to equations without any parameters at all. In fact even $\omega \subset M$ because each individual integer is also unique solution to such an equation. Therefore if $A \in M$ and A is countable then $A \subset M$: there must be a solution to "x is a surjection from ω onto A" in M, but then, again by uniqueness, $x(n) \in M$ for all n. This all shows that $\delta_M = M \cap \omega_1$ is a countable ordinal. Given $n \in \omega$ there is $\beta < \delta_M$ such that $|f(\gamma) - f(\delta_M)| < 2^{-n-1}$ for $\gamma \in [\beta, \delta_M]$. Let $\varphi(x, \beta, f)$ denote $x \in [\beta, \omega_1) \wedge |f(x) - f(\beta)| \geqslant 2^{-n}$. There is no solution to $\varphi^M(x, \beta, f)$ in M, hence there is none to $\varphi(x, \beta, f)$ in V. Therefore $|f(x) - f(\beta)| < 2^{-n}$ for all $x \geqslant \beta$. Combining this we find that f is constant on $[\delta_M, \omega_1)$ (and, in restrospect and by elementarity, even on $[\gamma, \omega_1)$ for some $\gamma \in M$).

Arkhangel'skiĭ's Theorem

Consider a first-countable compact Hausdorff space X; we wish to show that $|X| \leqslant \mathfrak{c}$. The proof sketched in the article *Cardinal functions I* is tailor-made for the elementary-submodel approach. One takes an elementary substructure M of the universe, of cardinality $\mathfrak{c}$ with $X \in M$ and such that all countable subsets of M are elements of M. One proves first that $X \cap M$ is closed in X: if $x \in \mathrm{cl}(X \cap M)$ then x is the limit of a sequence from $X \cap M$, this sequence is an element of M and hence so is its limit, i.e., $x \in M$. Second one proves that if $y \in X \setminus M$ then there is a finite family $\mathcal{O}$ of open sets *in M* that covers $X \cap M$ but with $y \notin \bigcup \mathcal{O}$ – this is possible because M must contain a countable local base at each point of $X \cap M$. But now $x \in X \setminus \bigcup \mathcal{O}$ has no solution in M, whereas it does have a solution in V.

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