e-7 Uniform Spaces, I

1. Definitions

Uniform spaces can be defined in various equivalent ways. We shall discuss the two principal methods and show how they are used to put uniform structures on ***metric spaces*** and ***topological groups***. Below the metric space referred to will be (X, d) and the topological group will be G and $\mathcal{N}$ denotes the ***neighbourhood filter*** of the neutral element e. A word on terminology: usually the uniform structure is simply called a **uniformity** – for expository purposes we use adjectives (diagonal, covering, pseudometric) to distinguish the various approaches.

Entourages
A **diagonal uniformity** on a set X is a ***filter*** $\mathcal{U}$ of subsets of $X \times X$ with the following properties.

(E1) $\Delta \subseteq U$ for all $U \in \mathcal{U}$;
(E2) if $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$; and
(E3) for every $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$.

Here $\Delta = \{(x, x)\colon x \in X\}$, the **diagonal** of X; $U^{-1} = \{(x, y)\colon (y, x) \in U\}$, the **inverse** of U; and $V^2 = \{(x, y)\colon (\exists z)((x, z), (z, y) \in V)\}$, the **composition** of V with itself. Each member of $\mathcal{U}$ is called an **entourage** of the diagonal.

For a metric space its **metric uniformity** is the filter generated by the sets $U_r = \{(x, y)\colon d(x, y) < r\}$.

For a topological group we have four natural filters. The **left uniformity** $\mathcal{U}_l$ is the filter generated by the sets $L_N = \{(x, y)\colon x^{-1}y \in N\}$, the **right uniformity** $\mathcal{U}_r$ is generated by the sets $R_N = \{(x, y)\colon yx^{-1} \in N\}$, the **two-sided uniformity** $\mathcal{U}_t$ generated by the sets $T_N = L_N \cap R_N$ and the fourth (nameless) uniformity $\mathcal{U}_s$ generated by the sets $S_N = L_N \cup R_N$.

Note that $\mathcal{U}_t$ is the filter generated by $\mathcal{U}_l \cup \mathcal{U}_r$ and that $\mathcal{U}_s = \mathcal{U}_l \cap \mathcal{U}_r$; also observe that if G is Abelian all four filters coincide.

A **base for a uniformity** $\mathcal{U}$ is nothing but a base for the filter $\mathcal{U}$. All the uniformities above were described by specifying bases for them. This explains why ***filter bases*** satisfying (E1)–(E3) above are sometimes called **uniformity bases**.

Uniform covers
A **covering uniformity** on a set X is a family $\mathfrak{U}$ of covers with the following properties.

(C1) If $\mathcal{A}, \mathcal{B} \in \mathfrak{U}$ then there is a $\mathcal{C} \in \mathfrak{U}$ that is a ***star refinement*** of both $\mathcal{A}$ and $\mathcal{B}$;
(C2) if a cover has a refinement that is in $\mathfrak{U}$ then the cover itself is in $\mathfrak{U}$.

These conditions say, in effect, that $\mathcal{U}$ is a ***filter*** in the partially ordered set of all covers, where the order is by star refinement. Each member of $\mathcal{U}$ is called a **uniform cover**.

A cover of a metric space is uniform if it is refined by $\{B_d(x, r)\colon x \in X\}$ for some $r > 0$.

In a topological group we get, as before, four types of uniform covers, each one defined by the requirement of having a refinement of the form $\mathcal{L}_N = \{xN\colon x \in G\}$, $\mathcal{R}_N = \{Nx\colon x \in G\}$, $\mathcal{T}_N = \{xN \cap Nx\colon x \in G\}$ or $\mathcal{S}_N = \{xN \cup Nx\colon x \in G\}$, respectively. Again, in an Abelian group, for each N, the covers $\mathcal{L}_N$, $\mathcal{R}_N$, $\mathcal{T}_N$ and $\mathcal{S}_N$ are identical.

A subfamily $\mathfrak{B}$ of a uniformity $\mathfrak{U}$ is a base if every element of $\mathfrak{U}$ has a refinement that is in $\mathfrak{B}$. Any family of covers that satisfies (C1) can serve as a base for some uniformity.

Equivalence of the approaches
From a diagonal uniformity $\mathcal{U}$ one defines a base for a covering uniformity: all covers of the form $\{U[x]\colon x \in X\}$ for some $U \in \mathcal{U}$.

Conversely from a covering uniformity $\mathfrak{U}$ one defines a base for a diagonal uniformity: all sets of the form $\bigcup\{A \times A\colon A \in \mathcal{A}\}$, where $\mathcal{A} \in \mathfrak{U}$.

These operations are each others inverses and establish an order-preserving bijection between the families of both kinds of uniformities.

Pseudometrics (Gauges)
Yet another way of introducing uniform structures is via ***pseudometrics*** or **gauges** as they are often called in this context.

If above, instead of a metric space, we had used a ***pseudometric space*** nothing would have changed. In fact, one can start with any family P of pseudometrics and define $U_d(r) = \{(x, y)\colon d(x, y) < r\}$ for $d \in P$ and $r > 0$. The resulting family $\{U_d(r)\colon d \in P,\ r > 0\}$ of entourages is a **subbase for a uniformity** in that the family of finite intersections is a base for a uniformity, denoted $\mathcal{U}_P$.

It is a remarkable fact that every uniformity has a subbase, even a base, of this form. Given a sequence $\{V_n\colon n \in \mathbb{N}\}$ of entourages on a set X such that $V_0 = X^2$ and $V_{n+1}^3 \subseteq V_n$ for all n one can find a pseudometric d on X such that $U_d(2^{-n}) \subseteq V_n \subseteq \{(x, y)\colon d(y, x) \leqslant 2^{-n}\}$ for all n [E, 8.1.10] (there is a similar theorem for ***normal sequences*** of covers [E, 5.4.H]). Thus every uniform structure can be defined by a family of pseudometrics. The family of all pseudometrics d that satisfy $(\forall r > 0)(U_d(r) \in \mathcal{U})$ is denoted $P_{\mathcal{U}}$; it is the largest family of pseudometrics that generate $\mathcal{U}$. The family $P_{\mathcal{U}}$ satisfies the following two properties.

(P1) if $d, e \in D$ then $d \vee e \in D$;

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(P2) if e is a pseudometric and for every $\varepsilon > 0$ there are $d \in P$ and $\delta > 0$ such that always $d(p,q) < \delta$ implies $e(p,q) < \varepsilon$ then $e \in P$.

A family P of pseudometrics with these properties is called a **pseudometric uniformity**; it satisfies the equality $P_{\mathcal{U}_P} = P$.

Uniform topology

Every uniform space carries a natural topology $\tau_{\mathcal{U}}$, its **uniform topology**. It is defined using ***neighbourhood bases***. Given a diagonal uniformity $\mathcal{U}$ one uses $\{U[x]\colon U \in \mathcal{U}\}$ for each x, in the case of a covering uniformity $\mathfrak{U}$ one uses $\{\mathrm{St}(x,\mathcal{A})\colon \mathcal{A} \in \mathfrak{U}\}$ and a pseudometric uniformity P provides $\{B_d(x,r)\colon d \in P,\ r > 0\}$.

Alternatively one could have used a ***closure operator***, for in the uniform topology one has $\mathrm{cl}\, D = \bigcap\{U[D]\colon U \in \mathcal{U}\} = \bigcap\{\mathrm{St}(D,\mathcal{A})\colon \mathcal{A} \in \mathfrak{U}\}$ for all subsets of X.

In general not all entourages are open, nor do all uniform covers consist of open sets but the open entourages and the open uniform covers do form bases for the uniform structures – this implies that every uniform cover is a ***normal cover***. Also, if U is an entourage then $\mathrm{cl}\, U \subseteq U^2$ so that the closed entourages form a base as well.

It is readily seen that for every pseudometric d in the family $P_{\mathcal{U}}$ and every x the map $y \mapsto d(x,y)$ is continuous with respect to $\tau_{\mathcal{U}}$; this establishes that the uniform topology is ***completely regular*** (possibly not ***Hausdorff***). The uniform topology is Hausdorff iff it is T_0 and this is the case if the uniformity is **separated** or **Hausdorff**, which means that $\bigcap \mathcal{U} = \Delta$ or, equivalently, $\{x\} = \bigcap\{\mathrm{St}(x,\mathcal{A})\colon \mathcal{A} \in \mathfrak{U}\}$ for all x.

The intersection $\equiv\, = \bigcap \mathcal{U}$ is an equivalence relation on the set X and the uniformity $\mathcal{U}$ can be transferred to a uniformity $\widehat{\mathcal{U}}$ on the set $\widehat{X} = X/{\equiv}$ to yield a separated uniform space, that to most intents and purposes is interchangeable with $(X,\mathcal{U})$.

A topological term applied to a uniform space usually refers to a property of the uniform topology, although ambiguity has crept in, see, e.g., the term ***weight*** below. Certain topological terms take the modifier 'uniformly'; it usually means that one entourage works for all points simultaneously. A **uniformly locally compact space** for instance has an entourage such that $U[x]$ is compact for all x; the locally compact ***ordinal space*** ω_1 is locally compact but not uniformly so.

A topological space $(X,\mathcal{T})$ is **uniformizable** if there is a (**compatible**) uniformity on X whose uniform topology is $\mathcal{T}$. Thus, a uniformizable topological space is completely regular; the converse is also true: associate to every real-valued function $f\colon X \to \mathbb{R}$ the pseudometric d_f defined by $d_f(x,y) = |f(x) - f(y)|$; the resulting family of pseudometrics generates a compatible uniformity.

Some natural questions

It makes topological sense to ask whether the open sets in a space can generate a uniformity.

The property that the family of all neighbourhoods of the diagonal forms a uniformity is called **divisibility**. It is a property shared by ***paracompact*** Hausdorff spaces and ***generalized ordered spaces*** and it implies ***collectionwise normality***.

The related property that the family of all open covers is a base for a uniformity characterizes ***fully normal spaces***.

2. Uniform properties

When trying to generalize metric concepts to wider classes of spaces one encounters the countability barrier: almost no non-trivial uncountable construction preserves metrizability. The category of uniform spaces and uniformly continuous maps provides a convenient place to carry out these generalizations.

Below we invariably let X be our uniform space, with $\mathcal{U}$ its family of entourages and $\mathfrak{U}$ the family of uniform covers.

Uniform continuity

A map $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ between uniform spaces is **uniformly continuous** if $(f \times f)^{-1}[V] \in \mathcal{U}$ whenever $V \in \mathcal{V}$, equivalently, if $\{f^{-1}[A]\colon A \in \mathcal{A}\} \in \mathfrak{U}$ whenever $\mathcal{A} \in \mathfrak{V}$. A uniformly continuous map is also continuous with respect to the uniform topologies and the converse is, as in the metric case, true for compact Hausdorff spaces.

A bijection that is uniformly continuous both ways is a **uniform isomorphism**. A **uniform property** then is a property of uniform spaces that is preserved by uniform isomorphisms.

Products and subspaces

It is straightforward to define a uniform structure on a subset Y of a uniform space X: simply intersect the entourages with $Y \times Y$ (or trace the uniform covers on Y). To define a **product uniformity** one may follow the construction of the ***product topology*** and define a subbase for it. Given a family $\{(X_i,\mathcal{U}_i)\}_{i \in I}$ of uniform spaces define a family of entourages in the square of $\prod_i X_i$ using the projections π_i: $\{(\pi_i \times \pi_i)^{-1}[U]\colon U \in \mathcal{U}_i,\ i \in I\}$.

These constructions have the right categorical properties, so that we obtain subobjects and products in the category of uniform spaces and uniformly continuous maps. The uniform topology derived from the product and subspace uniformities are the product and subspace topologies derived from the original uniform topologies, respectively.

Uniform quotients

A map $q\colon X \to Q$ between uniform spaces is a **uniform quotient map** if it is onto and has the following universality property: whenever $f\colon Q \to Y$ is a map to a uniform space Y then f is uniformly continuous if $f \circ q$ is. Every uniformly continuous map $f\colon X \to Y$ admits a factorization $f = f_0 \circ q$ with q a uniform quotient map and f_0 a (uniformly continuous) injective map.

In analogy with the topological situation one can, given a surjection f from a uniform space X onto a set Y, define the

quotient uniformity on Y to be the finest uniformity that makes f uniformly continuous. The resulting map is uniformly quotient and all uniform quotient maps arise in this way.

The uniform topology of a quotient uniformity is not always the quotient topology of the original uniform topology: if X is completely regular but not normal, as witnessed by the disjoint closed sets F and G, then identifying F to one point results in a space which is Hausdorff but not (completely) regular, hence the quotient uniformity from the fine uniformity (see below) does not generate the quotient topology. There is even a canonical construction that associates to every separated uniform space X a uniform space Y with a uniform quotient map $q : X \to Y$ and such that the uniform topology of Y is discrete, see [6, Exercise III.3].

Uniform quotient maps behave different from topological quotient maps in other respects as well: every product of uniform quotient maps is again a uniform quotient map [5].

Completeness
We say that X is **complete** (and $\mathcal{U}$ or $\mathfrak{U}$ a **complete uniformity**) if every **Cauchy filter** converges. A filter $\mathcal{F}$ is Cauchy if for every $V \in \mathcal{U}$ there is $F \in \mathcal{F}$ with $F \times F \subseteq V$ or, equivalently, if $\mathcal{F} \cap \mathcal{A} \neq \emptyset$ for all $\mathcal{A} \in \mathfrak{U}$. Closed subspaces and products of complete spaces are again complete.

Every uniform space has a **completion**, this is a complete uniform space that contains a dense and uniformly isomorphic copy of X. As underlying set of a completion one can take the set $\widetilde{X}$ of *minimal* Cauchy filters. Every entourage U of $\mathcal{U}$ is extended to $\widetilde{U} = \{(\mathcal{F}, \mathcal{G})\colon (\exists F \in \mathcal{F})(\exists G \in \mathcal{G})(F \times G \subseteq U)\}$; the family $\{\widetilde{U}\colon U \in \mathcal{U}\}$ generates a complete uniformity on $\widetilde{X}$. If $x \in X$ then its neighbourhood filter $\mathcal{F}_x$ is a minimal Cauchy filter and $x \mapsto \mathcal{F}_x$ is a uniform embedding.

As in the case of ***metric completion*** the completion of a separated uniform space is unique up to uniform isomorphism.

Using the canonical correspondence between nets and filters (see the article on *Convergence*) one can define a **Cauchy net** to be a ***net*** whose associated filter is Cauchy. This is equivalent to a definition more akin to that of a ***Cauchy sequence***: A net $(t_\alpha)_{\alpha \in D}$ is Cauchy if for every entourage U there is an α such that $(t_\beta, t_\gamma) \in U$ whenever $\beta, \gamma \geqslant \alpha$.

Total boundedness
We say X is **totally bounded** or **precompact** if for every entourage U (or uniform cover $\mathcal{A}$) there is a finite set F such that $U[F] = X$ (or $\mathrm{St}(F, \mathcal{A}) = X$). Subspaces and products of precompact spaces are again precompact.

A metrizable space has a compatible totally bounded metric iff it is ***separable***. A uniformizable space always has a compatible totally bounded uniformity; indeed, for any uniform space $(X, \mathcal{U})$ the family of all finite uniform covers is a base for a (totally bounded) uniformity $p\mathcal{U}$ with the same uniform topology, this uniformity is the **precompact reflection** of $\mathcal{U}$.

The metric theorem that equates compactness with completeness plus total boundedness remains valid in the uniform setting; likewise a Tychonoff space is compact if every compatible uniformity is complete. It is not true that a Tychonoff space is compact iff every compatible uniformity is totally bounded. The ordinal space ω_1 provides a counterexample: it is not compact and it has only one compatible uniformity (the family of all neighbourhoods of the diagonal), which necessarily is totally bounded.

Uniform weight
The **weight**, $w(X, \mathcal{U})$, is the minimum cardinality of a base. A uniformity $\mathcal{U}$ can be generated by κ pseudometrics iff $w(X, \mathcal{U}) \leqslant \kappa \cdot \aleph_0$ iff the separated quotient $\widehat{X}$ admits a uniform embedding into a product of κ many (pseudo)metric spaces (with its product uniformity). In particular: a uniformity is generated by a single pseudometric iff its weight is countable.

The **uniform weight** $u(X)$ of a Tychonoff space X is the minimum weight of a compatible uniformity. This is related to other cardinal functions by the inequalities $u(x) \leqslant w(X) \leqslant u(X) \cdot c(X)$. The first follows by considering a compactification of the same weight as X, the second from the fact that each pseudometric contributes a ***σ-discrete*** family of open sets to a base for the open sets. The uniform weight is related to the **metrizability degree**: $m(X)$ is the minimum κ such that X has an open base that is the union of κ many discrete families, whereas $u(X)$ is the minimum κ such that X has an open base that is the union of κ many discrete families of ***cozero sets***. Thus $m(X) \leqslant u(X)$; equality holds for normal spaces and is still an open problem for Tychonoff spaces.

3. Further topics

Fine uniformities
Every family $\{\mathcal{U}_i\}_i$ of uniformities has a supremum $\bigvee_i \mathcal{U}_i$. In terms of entourages it is generated by the family of all finite intersections of elements of $\bigcup_i \mathcal{U}_i$, i.e., $\bigcup_i \mathcal{U}_i$ is used as a subbase. If all the $\mathcal{U}_i$ are compatible with a fixed topology $\mathcal{T}$ then so is $\bigvee_i \mathcal{U}_i$. This implies that every Tychonoff space admits a finest uniformity, the **fine uniformity** or **universal uniformity**, it is the one generated by the family of *all* normal covers or by the family of all pseudometrics d_f defined above. The fine uniformity is denoted $\mathcal{U}_f$.

One says that a uniformity $\mathcal{U}$ itself is **fine** (or a **topologically fine uniformity**) if it is the fine uniformity of its uniform topology $\tau_{\mathcal{U}}$.

The equivalence of full normality and paracompactness combined with the constructions of pseudometrics described above yield various characterizations of the covers that belong to the fine uniformity: they are the covers that have ***locally finite*** (or ***σ-locally finite*** or ***σ-discrete***) refinements consisting of cozero sets. From this it follows that the precompact reflection of the fine uniformity is generated by the finite cozero covers.

Continuity versus uniform continuity
Every continuous map from a fine uniform space to a uniform space (or pseudometric space) is uniformly continuous;

this property characterizes fine uniform spaces. Uniform spaces on which every continuous real-valued function is uniformly continuous are called ***UC-spaces***. A *metric* UC-space is also called an **Atsuji space**.

The precompact reflection of a fine uniformity yields a space where all *bounded* continuous real-valued functions are uniformly continuous, these are also called ***BU-spaces***.

Compactifications
There is a one-to-one correspondence between the families of ***compactifications*** of a Tychonoff space and the compatible totally bounded uniformities. If γX is a compactification of X then the uniformity that X inherits from γX is compatible and totally bounded. Conversely, if $\mathcal{U}$ is a compatible totally bounded uniformity on X then its completion is a compactification of X, the **Samuel compactification** of $(X, \mathcal{U})$. The correspondence is order-preserving: the finer the uniformity the larger the compactification. Consequently the compactification that corresponds to the precompact reflection $\mathcal{U}_{\text{fin}}$ of the fine uniformity is exactly the ***Čech–Stone compactification***. It also follows that a space has exactly one compatible uniformity iff it is ***almost compact***.

Proximities
There is also a one-to-one correspondence between the ***proximities*** and precompact uniformities.

Indeed, a uniformity $\mathcal{U}$ determines a proximity $\delta_{\mathcal{U}}$ by $A\ \delta_{\mathcal{U}}\ B$ iff $U[A] \cap U[B] \neq \emptyset$ for every entourage U (intuitively: proximal sets have distance zero).

Conversely, a proximity δ determines a uniformity $\mathcal{U}_\delta$: the family of sets $X^2 \setminus (A \times B)$ with $A \not\delta B$ forms a subbase for $\mathcal{U}_\delta$. This uniformity is always precompact and, in fact, $\mathcal{U}_{\delta_{\mathcal{U}}}$ is the precompact reflection of $\mathcal{U}$.

The Samuel compactification of $(X, \mathfrak{U}_\delta)$ is also known as the **Smirnov compactification** of (X, δ).

Function spaces
Uniformities also allow one to formulate and prove theorems on uniform convergence and continuity in a general setting. Thus, given a uniform space $(Y, \mathcal{V})$ and a set (or space) X one can define various uniformities on the set Y^X of all maps from X to Y. Let $\mathcal{A}$ be a family of subsets of X. For $V \in \mathcal{V}$ and $A \in \mathcal{A}$ one defines the entourage $E_{A,V}$ to be the set $\{(f, g)\colon (\forall x \in A)((f(x), g(x)) \in V)\}$. The family of sets $E_{A,V}$ serves as a subbase for a uniformity. The corresponding uniform topology is called the **topology of uniform convergence** on members of $\mathcal{A}$.

If $\mathcal{A} = \{X\}$ then we obtain the topology of uniform convergence: a net $(f_\alpha)_\alpha$ converges with respect to this topology iff it **converges uniformly**: $f_\alpha \to f$ uniformly if for every $V \in \mathcal{V}$ there is an α_0 such that $(f_\alpha(x), f_{\alpha_0}(x)) \in V$ for all $x \in X$ and all $\alpha \geqslant \alpha_0$. One proves that uniform limits of (uniformly) continuous maps are again (uniformly) continuous, thus freeing these theorems from the bonds of countability.

If $\mathcal{A}$ is the family of finite subsets of X then one recovers the product uniformity and the ***topology of pointwise convergence***. If X is a topological space and $\mathcal{A}$ is the family of compact sets then the uniform topology, when restricted to the set $C(X, Y)$ of all continuous maps, is the ***compact-open topology***.

The combinatorics of uniform covers
It follows from the proof of the theorem that fully normal spaces are paracompact (see the article *Paracompact spaces*) that every uniform cover has a locally finite open (even cozero) refinement. The natural question whether this refinement may be chosen to be a uniform cover has a negative answer [8, 10]. Indeed, the metric uniformity of the Banach space $\ell_\infty(\aleph_1)$ provides a counterexample. A more general theorem can be formulated using some additional terminology. The **degree of a family** $\mathcal{A}$ is the minimum cardinal κ with the property that $|\mathcal{B}| < \kappa$ whenever $\mathcal{B} \subseteq \mathcal{A}$ and $\bigcap \mathcal{B} \neq \emptyset$. The **point-character of a uniform space** $(X, \mathfrak{U})$ is the minimum κ such that $\mathfrak{U}$ has a base consisting of covers of degree less than κ, it is denoted $pc(X, \mathfrak{U})$. Thus $pc(\ell_\infty(\aleph_1)) > \aleph_0$ and, in general, $pc(\ell_\infty(\lambda)) > \kappa$, whenever $\kappa < \lambda$ is regular, see [8].

In a uniform space $(X, \mathfrak{U})$ the finite uniform covers form a base for a uniformity, as do the countable uniform covers. The corresponding statement for higher cardinals is consistent with and independent of ZFC: for example, the ***Continuum Hypothesis*** implies that the uniform covers of cardinality $\aleph_1$ (or less) form a base for a uniformity, whereas ***Martin's Axiom*** implies that for $\ell_\infty(\aleph_1)$ this is not the case, see [8].

4. Completeness and completions

A Tychonoff space is **Dieudonné complete** or **topologically complete** or **completely uniformizable** if it has a compatible complete uniformity, equivalently, if the fine uniformity is complete.

Paracompact spaces are Dieudonné complete, indeed if a filter $\mathcal{F}$ does not converge then $\{X \setminus \operatorname{cl} F\colon F \in \mathcal{F}\}$ is an open cover, which belongs to the fine uniformity, so that $\mathcal{F}$ is not Cauchy.

Realcompact spaces are Dieudonné complete – the countable cozero covers generate a complete uniformity $\mathfrak{U}_\omega$ – and the converse is true provided the cardinality of the space (or better of its closed discrete subspaces) is not ***Ulam measurable*** – this is Shirota's theorem. The role of ***measurable cardinals*** is plain from the fact that a non-trivial countably complete ***ultrafilter*** is a Cauchy filter with respect to $\mathfrak{U}_\omega$ but not with respect to the fine uniformity, where we consider the measurable cardinal with its discrete topology.

Many books in General Topology provide introductions to uniform spaces; we mention Chapter 8 of [E], Chapter 7 of [3] and Chapter 15 of [4]; the latter deserves mention because it uses pseudometric exclusively. Isbell's book [6] is more comprehensive and spurred a lot of research in the years after its publication.

Page's book [7] concerns the workings of uniform spaces in topological groups and (Functional) Analysis; the monograph [9] by Roelcke and Dierolf treats topological groups from a uniform viewpoint; and Benyamini and Lindenstrauss' [2] offers more applications in the geometry of Banach spaces.

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