j-2 Consistency Results in Topology, I: Quotable Principles

A noteworthy feature of general topology – as contrasted with geometry, number theory, and most other fields of mathematics – is that many of its fundamental questions are not decided by "the usual axioms of set theory". What this means is that if one codifies the set-theoretic assumptions used implicitly in ordinary mathematics and consolidates these in a list of axioms – the usual axioms of set theory – one is left with many statements that can neither be proved nor refuted on the basis of these axioms alone. In some sense such undecidability results are commonplace: everyone knows that, say, the statement $\varphi = (\forall x)(\forall y)(xy = yx)$ cannot be decided by "the usual axioms of Group Theory". Indeed, the group of integers is a **model** for "Group Theory plus φ", which makes this conjunction consistent and φ non-refutable; the permutation group S_3 does the same for $\neg\varphi$.

A convenient list of axioms for Set Theory, one which has become standard is "ZFC" – Zermelo–Fraenkel set theory, including the Axiom of Choice. See, e.g., [Ku] for a detailed exposition. Thus there are important topological statements φ such that neither φ nor its negation follow from ZFC. Establishing this is not as easy as in the case of Group Theory. While it is easy to prove the consistency of Group Theory by providing models (groups) for it, the same is, by Gödel's Second Incompleteness Theorem, in principle impossible for ZFC (or any other useful collection of axioms for set theory). Gödel's theorem says that a consistency proof of a theory as strong as ZFC cannot be formalized in the theory itself. This explains why many results are formulated as "if ZFC is consistent then so is ZFC plus φ" instead of simply "φ is consistent with ZFC" and why, formally, we should be speaking of *relative* consistency proofs. Even if one is not interested in consistency results per se, it is nonetheless prudent to be aware of them, lest one waste effort trying to prove a proposition that has a consistent negation.

There are two ways in which consistency results in topology are obtained. The first consists of proving implications between statements, where the antecedent is a statement previously proven consistent with ZFC. Thus, the consequent itself is proven consistent with ZFC as well. The present article deals with results like these and in particular with the best-known combinatorial principles that occur as antecedents. In the second kind of consistency result will be dealt with in the companion article; here one needs an intimate knowledge of *how* one actually proves (relative) consistency results. The principal subjects in that article are forcing and large cardinals.

1. The Continuum Hypothesis

The best-known quotable principle is undoubtedly the **Continuum Hypothesis**, abbreviated as CH, which states that the real line has minimum possible cardinality, i.e., $\mathfrak{c} = |\mathbb{R}| = 2^{\aleph_0} = \aleph_1$. It was proved consistent by Gödel in [5] (see [Ku] for a proof) but before that numerous consequences were derived from it, see, e.g., [7].

It derives its strength from the facts that so many sets have cardinality $\mathfrak{c}$ and the initial segments of ω_1 are all countable; this makes for relatively easy transfinite constructions. To give the flavour we construct a **Lusin set**: an uncountable subset of $\mathbb{R}$ that meets every nowhere dense set in a countable set. It suffices to take an enumeration $\langle N_\alpha \colon \alpha < \omega_1 \rangle$ of the family of all closed nowhere dense sets and apply the ***Baire Category Theorem*** each time to choose l_α outside $\bigcup_{\beta<\alpha} N_\alpha$; then $\{l_\alpha \colon \alpha < \omega_1\}$ is the Lusin set. A similar construction will yield a ***strongly infinite-dimensional*** subspace of the ***Hilbert cube*** all of whose finite-dimensional subsets are countable. In [KV, Chapter 8, § 4] one finds more intricate constructions, involving CH, of ***hereditarily separable*** non***Lindelöf*** spaces (**S-spaces**) as well as ***hereditarily Lindelöf*** non***separable*** spaces (**L-space**s).

On occasion CH helps simply by counting: in combination with ***Jones' Lemma*** CH (or even its consequence $2^{\aleph_0} < 2^{\aleph_1}$) implies that in a separable normal space closed and discrete subsets are countable and hence that separable normal ***Moore space***s are metrizable. By contrast, CH implies there is a non-metrizable normal Moore space [KV, Chapter 16].

Another counting example comes from the theory of ***uniform spaces***. It is well-known that in a uniform space the finite uniform covers generate a uniformity again as do the countable uniform covers. The proofs can be generalized to show that any $\aleph_1$-sized uniform cover has a uniform star refinement of cardinality $\mathfrak{c}$. Thus, CH implies that the $\aleph_1$-sized uniform covers generate a uniformity. In general, the **Generalized Continuum Hypothesis** (GCH), which states that for all cardinals κ one has $2^\kappa = \kappa^+$, implies that for every κ the uniform covers of cardinality less than κ generate a uniformity.

The Continuum Hypothesis also implies that there is an almost disjoint family of uncountable subsets of ω_1 of cardinality $2^{\aleph_1}$ or, equivalently, that the ***cellularity*** of the space $U(\omega_1)$ of uniform ultrafilters is the maximum possible $2^{\aleph_1}$. Here 'almost disjoint' means that intersections are countable and an ultrafilter u on a set X is **uniform** if every element has full cardinality, i.e., $|U| = |X|$ for all $U \in u$.

See also the article on $\beta\mathbb{N}$ and $\beta\mathbb{R}$ for other applications of CH.

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2. Martin's Axiom

Every consequence of CH is a potential theorem, as it can no longer be refuted. Some consequences are in fact equivalent to CH, e.g., the existence of the special infinite-dimensional space above; such consequences are then automatically nonprovable because Cohen proved the consistency of the negation of CH (see the next article for an indication of the proof and [Ku] for a full treatment). To decide other consequences one would need proofs that avoid CH or principles that imply their negations.

Martin's Axiom (MA) is such a principle. It can be viewed as an extension of the Baire Category Theorem: it states that if X is a ***compact Hausdorff*** space with the ***countable chain condition*** (the ***ccc*** for short) then the union of fewer than $\mathfrak{c}$ many nowhere dense in X still has a dense complement. As such it is a consequence of CH but the conjunction MA $+ \neg$CH is also consistent.

Its original formulation, though more extensive, is ultimately more useful: if $\mathbb{P}$ is a partially ordered set (a poset) with the ccc and $\mathcal{D}$ is a family of fewer than $\mathfrak{c}$ dense sets then there is a filter G on $\mathbb{P}$ that meets all members of $\mathcal{D}$. Two elements p and q of $\mathbb{P}$ are **compatible** if there is an element r with $r \leqslant p, q$, and **incompatible** otherwise. An **antichain** is a set of mutually incompatible elements and "$\mathbb{P}$ has the ccc" means every antichain is countable. A set $D \subseteq \mathbb{P}$ is a **dense set** if for every p there is $d \in D$ with $d \leqslant p$. Finally, a **filter** on $\mathbb{P}$ is a subset G that satisfies: if $p, q \in G$ then there is $r \in G$ with $r \leqslant p, q$, and if $p \in G$ and $q \geqslant p$ then $q \in G$.

The equivalence between the two formulations becomes somewhat apparent when one thinks of $\mathbb{P}$ as representing the open sets of the space X – a proof may be devised along the lines of the ***Stone Representation Theorem*** of Boolean algebras. The usefulness of the poset formulation may be illustrated by a proof of the equality $2^{\aleph_0} = 2^{\aleph_1}$ from MA $+ \neg$CH. We shall construct an injective map from the power set $\mathcal{P}(\omega_1)$ of ω_1 into $\mathcal{P}(\mathbb{N})$, using an ***almost disjoint family*** $\{x_\alpha : \alpha < \omega_1\}$ on $\mathbb{N}$. Given $A \subseteq \omega_1$ we find $B_A \subseteq \mathbb{N}$ satisfying "$B_A \cap x_\alpha$ is infinite iff $\alpha \in A$", which makes $A \mapsto B_A$ one-to-one. For this we use a poset of approximations to B_A. An element of $\mathbb{P}$ is a ordered pair $p = \langle F_p, b_p \rangle$, where F is a finite subset of $\omega_1 \setminus A$ and b_p a finite subset of $\mathbb{N}$. We say $p \leqslant q$ if $F_p \supseteq F_q$, $b_p \supseteq b_q$ and $b_p \setminus b_q \cap \bigcup_{\alpha \in F_q} x_\alpha = \emptyset$. We interpret b_p as an approximation of B_A, with the promise that $B_A \cap x_\alpha = b_p \cap x_\alpha$ for $\alpha \in F_p$. For every $\alpha \in \omega_1 \setminus A$ the set $D_\alpha = \{p : \alpha \in F_p\}$ is dense in $\mathbb{P}$, as is $E_{\alpha,n} = \{p : |b_p \cap x_\alpha| > n\}$ for each $\alpha \in A$ and $n \in \mathbb{N}$. To see that $\mathbb{P}$ has the ccc, note that two elements with the same second coordinate are compatible: $\langle F_p \cup F_q, b \rangle \geqslant \langle F_p, b \rangle, \langle F_q, b \rangle$. Finally then if G is a filter that meets the dense sets above then $B_A = \bigcup \{b_p : p \in G\}$ is the required set.

The conjunction MA $+ \neg$CH has often been advertised as 'an alternative to the Continuum Hypothesis' and indeed, many consequences of CH become false if MA $+ \neg$CH is assumed. On the other hand, 'small cardinals' – those smaller than $\mathfrak{c}$ – behave like $\aleph_0$ under MA, e.g., sets of reals of size less than $\mathfrak{c}$ are ***meager*** and of measure zero. We discuss the consequences of CH mentioned above.

Lusin sets no longer exist as every set of reals of cardinality less than $\mathfrak{c}$ is of first category. It becomes harder to find S- and L-spaces. Indeed, MA $+ \neg$CH denies the existence of compact such spaces and it is even consistent with MA $+ \neg$CH that no ***S-spaces*** exist.

On the positive side: MA $+ \neg$CH implies that separable normal nonmetrizable Moore spaces exist. It also implies that there is a uniform space with a uniform cover of cardinality $\aleph_1$ without an $\aleph_1$-sized uniform star refinement, to wit the subspace $\{x : \|x\| = 1\ (\forall \alpha)(x_\alpha \geqslant 0)\}$ of $\ell_\infty(\aleph_1)$ [6]. This paper does not mention MA $+ \neg$CH directly but the proof needs a family $\mathcal{A}$, of size $\aleph_2$, of uncountable subsets of ω_1 such that for some fixed cardinal $\kappa \leqslant \aleph_2$ every subfamily $\mathcal{A}'$ of $\mathcal{A}$ of size κ has a finite intersection. To make such a family one starts with an almost disjoint family $\mathcal{B}$, of size $\aleph_2$, of uncountable subsets of ω_1 (almost disjoint means distinct elements have a countable intersection); by repeated application of [4, 42I] one shrinks the elements of $\mathcal{B}$ to produce the desired family $\mathcal{A}$ (with $\kappa = 2$).

The ccc is the weakest in a line of properties, the best-known of these are σ-centered (corresponding to separable compact spaces) and countable (corresponding to compact metrizable spaces). These in turn give rise to weakenings of MA that have generated interest of their own, since they provide the possibility of denying some of MA's consequences while retaining others.

Martin's Axiom for σ-centered partially ordered sets was shown to be equivalent to the purely combinatorial statement known variously as $P(\mathfrak{c})$ or $\mathfrak{p} = \mathfrak{c}$: if $\mathcal{F}$ is a family, of cardinality less than $\mathfrak{c}$, of subsets of $\mathbb{N}$ with the **strong finite intersection property**, i.e., the intersection of every finite subfamily is infinite, then there is an infinite subset A of $\mathbb{N}$ with $A \setminus F$ finite for all $F \in \mathcal{F}$.

Martin's Axiom for countable partially ordered sets is equivalent to the strong Baire Category Theorem for $\mathbb{R}$: if $\mathcal{U}$ is a family of fewer than $\mathfrak{c}$ dense open sets in $\mathbb{R}$ then $\bigcap \mathcal{U}$ is dense.

Another way to vary Martin's Axiom is to restrict the number of dense sets. Thus MA($\aleph_1$) means MA for families of $\aleph_1$ many dense sets. This version is strong enough to ensure there are no ***Souslin tree***s (see below for the definition).

Fremlin's book [4] and Weiss' survey [KV, Chapter 19] are good places to start exploring the consequences and versions Martin's Axiom.

3. The proper forcing axiom

The **Proper Forcing Axiom** (PFA) is a strengthening of MA, and a considerable one at that. Its formulation is quite similar, replacing 'ccc' by 'proper' and 'fewer than $\mathfrak{c}$' by '$\aleph_1$ many'. The notion of a **proper partial order** is more involved than that of a ccc partial order; it was developed in connection with iterations of forcing, for which see part II of this article. Given a set X, a subset of $[X]^{\aleph_0}$ (the family of countable subsets of X) is a **closed and unbounded set** if it is cofinal and closed under unions of countable chains. A **stationary set** is one that intersects every closed and unbounded set. A proper poset preserves stationary sets, i.e., $\mathbb{P}$ is proper means that for every set X and every stationary set in $[X]^{\aleph_0}$ remains stationary in any ***generic extension*** by $\mathbb{P}$ – this is not a given: $[X]^{\aleph_0}$ will most likely grow and so will the family of closed and unbounded sets. There is a combinatorial characterization of properness in terms of games: player I starts with $p \in \mathbb{P}$ and at move n chooses a maximal antichain A_n in $\mathbb{P}$ while player II the counters by choosing a countable subset B_i^n of A_i for each $i \leqslant n$. In the end we look at $B_i = \bigcup_{n \geqslant i} B_i^n$ and declare II the winner is there is a $q \leqslant p$ such that every B_i is **predense** below q, i.e., every $r \leqslant q$ is compatible with an element of B_i. The partial order $\mathbb{P}$ is proper precisely when II has a winning strategy for this game.

From this it follows easily that ccc partial orders are proper – simply take $B_n^n = A_n$ and $q = p$ – as are **countably closed** partial orders: at move n II picks p_n and $a_n \in A_n$ with $p_n \leqslant a_n$ (and $p_n \leqslant p_{n-1}$ when $n \geqslant 1$), she then plays $B_n^n = \{a_n\}$. In the end there is a q below all p_n by countable closedness; this q witnesses II's victory. The definition of properness implies that the forcing composition (and indeed iteration) of proper partial orders is again proper and in practice this is often how applications of PFA go. One has a candidate partial order for the problem at hand; this partial order is usually not proper, but after some preparatory forcing one can get a better version (even ccc) of the candidate. This preparation is itself often countably closed, so that the composition is proper – one applies PFA to this composition. The foregoing discussion should make clear that working with PFA is more involved than applying MA. The results obtained from PFA are generally much stronger than those obtainable from MA. Among (many) others, PFA implies: there are no S-spaces, every compact space of ***countable tightness*** is ***sequential***, there are no $\aleph_2$-Aronszajn trees. This last result implies that PFA harbours ***large cardinal*** strength, as the nonexistence of $\aleph_2$-Aronszajn trees implies $\aleph_2$ is a ***weakly compact cardinal*** in L (Gödel's constructible universe).

As it stands PFA implies MA($\aleph_1$) but it actually implies the full MA *because it implies* $2^{\aleph_0} = \aleph_2$, see [1].

Baumgartner's survey [KV, Chapter 21] is recommended as a first introduction to the Proper Forcing Axiom. Even stronger forcing axioms are coming into prominence: Martin's Maximum [3] and $\mathbf{P}_{\max}$ [9].

4. The Diamond principle

This is the first in a range of so-called prediction principles. It is denoted by $\diamondsuit$ and it states that there is a sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ (a **$\diamondsuit$-sequence**) of sets such that $A_\alpha \subseteq \alpha$ for all α and for every subset A of ω_1 the set $\{\alpha : A \cap \alpha = \alpha\}$ is stationary. This is intended to capture the essence of Jensen's proof that there is a **Souslin tree** in Gödel's constructible universe L (see [Ku, VII, B9]).

For much more on what follows, see [KV, Chapter 6]. A **tree** is a partially ordered set T in which for every x the set $\hat{x} = \{y : y \prec x\}$ of predecessors is well-ordered. The αth level of T is the set of x for which $\hat{x}$ has order type α. A **branch** in a tree is a maximal chain and an **antichain** is a set of mutually incomparable elements. A **κ-tree** is a tree of height κ (i.e., $T_\kappa = \emptyset$ and $T_\alpha \neq \emptyset$ for $\alpha < \kappa$) all of whose levels have cardinality less than κ. An **Aronszajn tree** is an $\aleph_1$-tree without branches of length ω_1 and a Souslin tree is an Aronszajn tree with no uncountable antichains. A **Kurepa tree** is an $\aleph_1$-tree with more than $\aleph_1$ branches of length ω_1. From a Souslin tree one can make a **Souslin line**, i.e., a linearly ordered set that is ccc but not separable in its order-topology and vice versa; thus, a Souslin tree/line refutes **Souslin's hypothesis**. We have indicated above that MA + ¬CH denies the existence of Souslin trees, in fact it implies that Aronszajn trees are **special**, which means that they can be covered by countably many antichains. The definitions of Aronszajn and Souslin trees carries over easily to larger cardinal numbers: $\aleph_1$ is replaced by the desired κ and 'countable' by 'smaller than κ'.

The construction of a Souslin tree is still one of the best introductions to the use of $\diamondsuit$. One constructs a tree-order $\prec$ on ω_1 in such a way that the interval $I_\alpha = [\omega \cdot \alpha, \omega \cdot (\alpha + 1))$ becomes the α-th level of the tree. Thus, $I_0 = [0, \omega)$ is left totally unordered. If $\alpha = \beta + 1$ then I_α is unordered but it supplies two direct $\prec$-successors for each point of I_β. If α is a limit and A_α is a maximal antichain in the ordering on $[0, \omega \cdot \alpha)$ constructed thus far then we choose countably many branches that cover the set and such that each passes through a point of A_α; we put the points of I_α on top of these branches (one point for each branch) – this ensures that A_α is even maximal in the final tree. In the end if A is a maximal antichain in the tree $(\omega_1, \prec)$, then the set of those α with $\alpha = \omega \cdot \alpha$ *and* $A \cap \alpha$ is maximal in $(\alpha, \prec)$ is closed and unbounded. There is therefore such an α with $A \cap \alpha = A_\alpha$, but A_α was to remain maximal, hence $A = A_\alpha$ and so A is countable.

One uses $\diamondsuit$ if the property under consideration allows **reflection**, as in the case above, where a maximal antichain intersects initial segments of the tree in maximal antichains – the $\diamondsuit$-sequence enables one to capture and deal with such reflections. There have been strengthenings of $\diamondsuit$ that assert that more is captured or more often; these are denoted by $\diamondsuit^*$, $\diamondsuit^+$, $\diamondsuit$ for stationary systems, etc. The last mentioned version implies that normal and first-countable spaces are $\aleph_1$-collectionwise Hausdorff, see [KV, Chapter 15].

It is clear that $\diamondsuit$ implies CH and, in fact $\diamondsuit = \mathsf{CH} + \clubsuit$, where $\clubsuit$ is a weakening of $\diamondsuit$: there is a sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ (a **$\clubsuit$-sequence**) such that S_α is a cofinal ω-sequence in $\omega \cdot \alpha$ and every uncountable subset of ω_1 contains some S_α. The $\clubsuit$-principle can be used to construct an S-space topology on ω_1 – make sure each initial segment is open and that $[\omega \cdot \alpha, \omega_1) \subseteq \operatorname{cl} S_\alpha$ – and a simple ***Dowker space*** on $\omega_1 \times \omega$ [KV, Chapter 17]. From the stronger $\diamondsuit$ one gets more: ***Ostaszewski's space*** and Fedorchuk's compact S-space of cardinality $2^{\mathfrak{c}}$ with no convergent sequences.

5. The open colouring axiom

The **Open Colouring Axiom** (OCA) is a Ramsey-type statement. It states that given a ***separable metrizable space*** X and an open subset K_0 of $[X]^2$ (the two-element subsets of X with the ***Vietoris topology***) there either is an uncountable subset Y of X with $[Y]^2 \subseteq K_0$ or $X = \bigcup_n X_n$ with $[X_n]^2 \cap K_0 = \emptyset$ for all n, see [8, Chapter 8].

It has a remarkable effect on the theory of the space $\mathbb{N}^*$. Very few of the results on this space proved from CH remain when OCA is assumed. Its status as a universal space is simply demolished: the ***Stone space*** of the ***measure algebra***, the square $\mathbb{N}^* \times \mathbb{N}^*$, the space $\mathbb{R}^*$, and many others are no longer continuous images of $\mathbb{N}^*$. The conjunction $\mathsf{OCA} + \mathsf{MA}$ implies that all autohomeomorphisms of $\mathbb{N}^*$ are induced by bijections between cofinite subsets of $\mathbb{N}$. The proofs of these results follow a by now well-established pattern: one proves that a potential map cannot have too much structure and one also proves that OCA implies (invariably via its second alternative) that a potential map must have a lot of structure.

A purely topological application of OCA is the following: if X is a cometrizable space then either X has a countable ***network***, or an uncountable discrete subspace, or it contains an uncountable subspace of the ***Sorgenfrey line***. A space is **cometrizable** if there is a weaker ***metrizable*** topology on it such that each point has a neighbourhood base consisting of sets which are closed in the metric topology

The memoir [2] contains many results related to OCA and gives lots of historic information.

In conclusion

We have barely scratched the surface of the use of quotable principles in General Topology; the volumes [KV] and [HvM] contain many applications of such principles.

A few words on the consistency of the principles. As mentioned above, CH holds in Gödel's Constructible Universe as do $\diamondsuit$ and its strengthenings. There is no such canonical model for the other principles discussed. The consistency of Martin's Axiom was established using the method of ***iterated forcing***, as was the consistency of PFA, though the latter required a ***supercompact cardinal*** in the initial model. The Open Colouring Axiom follows from PFA but its consistency may be established in an 'ordinary' iteration eschewing large cardinals.

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