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HEREDITARY INDECOMPOSABILITY AND THE INTERMEDIATE VALUE THEOREM

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Abstract. We show that hereditarily indecomposable spaces can be characterized by a special instance of the Intermediate Value Theorem in their ring of continuous functions.

Introduction

The classical Intermediate Value Theorem (IVT for short) states that if f is a continuous function from the interval $[a,b]$ to $\mathbb{R}$ with $f(a)\cdot f(b)<0$ then there is c in (a,b) such that $f(c)=0$. In [2] Henriksen, Larson and Martinez investigated forms of this theorem in lattice-ordered rings, where, because of the absence of any natural topology, they restricted their attention to polynomials. We mention some of their results for the ring $C^*(X)$ of bounded real-valued continuous functions on the topological space X; let us call X an IVT-space if the ring $C^*(X)$ satisfies the Intermediate

Value Theorem (the precise formulation of the IVT in this context follows below). The results are:

(1) every IVT-space is an F-space;
(2) every compact and zero-dimensional F-space is an IVT-space;
(3) every compact IVT-space is hereditarily indecomposable.

In this note we establish a partial converse to this last result in that we show that every compact hereditarily indecomposable space satisfies the IVT for a restricted class of polynomials.

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1. Preliminaries

We shall only deal with rings of the form $C^*(X)$, so we can, for the time being restrict our attention to compact Hausdorff spaces.

1.1. **The intermediate value theorem.** In the ring $C^*(X)$ the IVT takes on the following form: Let p be a polynomial with coefficients in $C^*(X)$ and let u and v be elements of $C^*(X)$ such that $p(u) \leqslant \mathbf{0} \leqslant p(v)$, where $\mathbf{0}$ denotes the zero function. Then there is $w \in C^*(X)$ such that $u \wedge v \leqslant w \leqslant u \vee v$ and $p(w) = \mathbf{0}$. The reason for working with $u \wedge v$ and $u \vee v$ is of course that it is usually not the case that $u(x) \leqslant v(x)$ for all x (or $v(x) \leqslant u(x)$ for all x).

To get some feeling for what the IVT says in this context let $p \in C^*(X)[t]$, so $p(t) = \sum_{i=0}^n f_i t^i$ for some elements $f_0, \ldots, f_n$ of $C^*(X)$, and let $u, v \in C^*(X)$ be such that $p(u) \leqslant \mathbf{0} \leqslant p(v)$. For every separate $x \in X$ we get an ordinary polynomial $p_x(t) = \sum_{i=0}^n f_i(x) t^i$; and the assumptions on u and v imply that $p\bigl(u(x)\bigr) \leqslant 0 \leqslant p\bigl(v(x)\bigr)$. The classical IVT therefore guarantees that there is a function $w : X \to \mathbb{R}$ such that $u \wedge v \leqslant w \leqslant u \vee v$ and $p(w) = \mathbf{0}$; the IVT for $C^*(X)$ demands that this w be continuous.

That this puts severe restrictions on the space X may be seen as follows: let $f \in C^*(X)$ and consider the polynomial $p(t) = |f|t - f$. Now $p(\mathbf{1}) = |f| - f \geqslant 0$ and $p(-\mathbf{1}) = -|f| - f \leqslant \mathbf{0}$, so if X is an IVT-space there must be a continuous function w such that $-\mathbf{1} \leqslant w \leqslant \mathbf{1}$ and $f = w|f|$. This however is one of the characterizations of F-spaces — see Gillman and Jerison [1].

1.2. **Hereditarily indecomposable spaces.** Much of what follows is taken from Oversteegen and Tymchatyn [3], which is a convenient survey on hereditarily indecomposable spaces.

To begin we recall that a continuum is said to be *indecomposable* if it cannot be written as the union of two proper subcontinua; it is *hereditarily indecomposable* if every subcontinuum is indecomposable.

We use the following characterization of hereditarily indecomposable continua.

Theorem 1.1. *A continuum X is hereditarily indecomposable if and only if whenever two disjoint closed sets A and B and open neighbourhoods U and V respectively are given we can write X as the union of three closed sets X_0, X_1 and X_2 such that $A \subseteq X_0$, $B \subseteq X_2$, $X_0 \cap X_1 \subseteq V$, $X_0 \cap X_2 = \varnothing$, and $X_1 \cap X_2 \subseteq U$.*

The property in this theorem can also be used to characterize those compact spaces (connected or not) for which every closed connected subspace is indecomposable; we shall call these compact spaces hereditarily indecomposable as well.

Observe that with this definition compact zero-dimensional spaces are hereditarily indecomposable as well.

2. The IVT implies hereditary indecomposability

In this section we reprove Theorem 3.2 from Henriksen, Larson and Martinez [2], which states that compact IVT-spaces are hereditarily indecomposable. In their proof these authors used a polynomial of degree 7 with two potentially irreducible quadratic factors. We use a completely factored polynomial of degree 3.

Theorem 2.1. *Compact IVT-spaces are hereditarily indecomposable.*

Proof. Let X be a compact IVT-space. To show that X is hereditarily indecomposable we take disjoint closed sets A and B and open sets U and V such that $A \subseteq U$ and $B \subseteq V$. We must exhibit three closed sets X_0, X_1 and X_2 such that $A \subseteq X_0$, $B \subseteq X_2$, $X_0 \cap X_1 \subseteq V$, $X_0 \cap X_2 = \varnothing$, $X_1 \cap X_2 \subseteq U$ and $X_0 \cup X_1 \cup X_2 = X$.

Choose a continuous function $f : X \to [0,1]$ such that $f \restriction A \equiv 0$, $f \restriction B \equiv 1$, $f^{-1}\bigl[[0,\frac{1}{2})\bigr] \subseteq U$ and $f^{-1}\bigl[(\frac{1}{2},1]\bigr] \subseteq V$. Using f we define three continuous functions, f_1, f_2 and f_3, as follows: first

$$f_1(x) = \begin{cases} f(x) - \frac{1}{4} & \text{if } f(x) \leqslant \frac{1}{4} \\ 0 & \text{if } \frac{1}{4} \leqslant f(x) \leqslant \frac{3}{4} \\ f(x) - \frac{3}{4} & \text{if } \frac{3}{4} \leqslant f(x); \end{cases}$$

second

$$f_2(x) = \begin{cases} \frac{1}{2}\bigl(f(x) - \frac{1}{4}\bigr) & \text{if } f(x) \leqslant \frac{1}{4} \\ 2\bigl(f(x) - \frac{1}{4}\bigr) & \text{if } \frac{1}{4} \leqslant f(x) \leqslant \frac{3}{4} \\ \frac{1}{2}\bigl(f(x) + \frac{5}{4}\bigr) & \text{if } \frac{3}{4} \leqslant f(x); \end{cases}$$

and third

$$f_3(x) = \begin{cases} f(x) + \frac{3}{4} & \text{if } f(x) \leqslant \frac{1}{4} \\ 1 & \text{if } \frac{1}{4} \leqslant f(x) \leqslant \frac{3}{4} \\ f(x) + \frac{1}{4} & \text{if } \frac{3}{4} \leqslant f(x); \end{cases}$$

(At this point the reader may find it instructive to draw the graphs of f_1, f_2 and f_3 in case $X = [0,1]$ and $f(x) = x$. The zig-zag that appears when one follows the graph of f_3 left-to-right until it meets the graph of f_2 then follows the graph of f_2 right-to-left until it meets the graph of f_1 and finally the graph of f_1 left-to-right until the end is characteristic of hereditarily indecomposable spaces.) Note that $f_1 \leqslant f_2 \leqslant f_3$.

Consider the polynomial p defined by $p(t) = (t - f_1)(t - f_2)(t - f_3)$. Then $p(\mathbf{0}) \leqslant \mathbf{0} \leqslant p(\mathbf{1})$, for one readily checks that

- $f_2(x) < 0 < f_3(x) < 1$ if $f(x) < \frac{1}{4}$;
- $f_1(x) = 0$ and $f_3(x) = 1$ if $\frac{1}{4} \leqslant f(x) \leqslant \frac{3}{4}$ and
- $0 < f_1(x) < 1 < f_2(x)$ if $f(x) > \frac{3}{4}$.

An application of the Intermediate Value Theorem gives us a continuous function $w : X \to [0,1]$ such that $p(w) = \mathbf{0}$.

Let $X_0 = \bigl\{x : w(x) = f_3(x)\bigr\}$, $X_1 = \{x : w(x) = f_2(x)\bigr\}$ and $X_2 = \bigl\{x : w(x) = f_1(x)\bigr\}$. We check that these sets have all the required properties.

- $X_0 \cup X_1 \cup X_2 = X$ because $p(w) = \mathbf{0}$;
- $A \subseteq X_0$ because if $x \in A$ then $f(x) = 0$, hence $w(x) = f_3(x)$;
- $B \subseteq X_2$ because if $x \in B$ then $f(x) = 1$, hence $w(x) = f_1(x)$;
- $X_0 \cap X_1 \subseteq V$ because if $x \in X_0 \cap X_1$ then $f_3(x) = w(x) = f_2(x)$ hence $f(x) = \frac{3}{4}$ and $x \in V$;
- $X_1 \cap X_2 \subseteq U$ because if $x \in X_1 \cap X_2$ then $f_1(x) = w(x) = f_2(x)$ hence $f(x) = \frac{1}{4}$ and $x \in U$ and
- $X_0 \cap X_2 = \varnothing$ because $f_3 - f_1 = \mathbf{1}$.

We conclude that X is indeed hereditarily indecomposable. □

As announced before, in the next section we shall see that hereditary indecomposability is in fact characterized by the particular instance of the Intermediate Value Theorem that was actually employed.

3. Hereditary indecomposability implies part of the IVT

In this section we show that every compact hereditarily indecomposable F-space X satisfies the Intermediate Value Theorem for *completely factored polynomials*, that is, polynomials that can be written as $\prod_{i=1}^{n}(t - f_i)$, where the f_i are elements of $C(X)$.

This is a rather limited class of polynomials of course but, as we saw in Section 2, the case $n = 3$ is already strong enough to imply hereditary indecomposability. The Intermediate Value Theorem for this class of polynomials therefore characterizes hereditary indecomposability for F-spaces.

So let X be a hereditarily indecomposable F-space and let p, defined by $p(t) = \prod_{i=1}^{n}(t - f_i)$, be a completely factored polynomial in $C(X)$. Assume furthermore that $u, v \in C(X)$ are such that $p(u) \leqslant \mathbf{0} \leqslant p(v)$. Through a series of reductions we show that there is $w \in C(X)$ such that $p(w) = \mathbf{0}$ and $u \wedge v \leqslant w \leqslant u \vee v$.

Lemma 3.1. *We may assume that* $f_1 \leqslant f_2 \leqslant \cdots \leqslant f_n$.

Proof. For each $i \leqslant n$ define g_i by

$$g_i = \bigwedge_{|F|=i} \bigvee_{j\in F} f_j.$$

Observe that $g_1 \leqslant g_2 \leqslant \cdots \leqslant g_n$ and that, for each individual x, the sets of values $\{g_1(x), g_2(x), \ldots, g_n(x)\}$ and $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ are equal. It follows from this that the coefficients of t^0, t^1, …, t^{n-1} in $\prod_{i=1}^{n}(t - f_i)$ and $\prod_{i=1}^{n}(t - g_i)$ are the same and hence that the polynomials are the same. □

The case $n = 1$ should offer no problems and the case $n = 2$ is dealt with in the following proposition, which is a special case of Theorem 2.3 (b) of Henriksen, Larson and Martinez [2]. In fact, the polynomial p need not even be factored; it can always be factored by completing the square.

Proposition 3.2. *Every space satisfies the Intermediate Value Theorem for monic quadratic polynomials.*

Proof. Let $p(t) = t^2 + 2ft + g$ be such a polynomial and assume that there are u and v such that $p(u) \leqslant \mathbf{0} \leqslant p(v)$. Completing the square gives us $q(t) = (t + f)^2 + g - f^2$. Now because $p(u) \leqslant \mathbf{0} \leqslant p(v)$ we know that $f^2 - g \geqslant \mathbf{0}$ so that we can write $f^2 - g = h^2$ for some nonnegative $h \in C(X)$. We find that $p(t) = (t + f - h)(t + f + h)$; write $-f - h = f_1$ and $-f + h = f_2$.

Observe that for each x either $v(x) \leqslant f_1(x)$ or $v(x) \geqslant f_2(x)$ and that $f_1(x) \leqslant u(x) \leqslant f_2(x)$. We cover our space by three closed sets: $P = \mathrm{cl}\{x : u(x) < v(x)\}$, $Q = \{x : u(x) = v(x)\}$ and $R = \mathrm{cl}\{x : u(x) > v(x)\}$. We now note that $u \leqslant f_2 \leqslant v$ on P (because $u(x) \leqslant f_2(x) \leqslant v(x)$ whenever $u(x) < v(x)$) and that $v \leqslant f_1 \leqslant u$ on R. We define w as the combination

$$(f_2 \restriction P) \triangledown (u \restriction Q) \triangledown (f_1 \restriction R).$$

Note that w is well-defined because, by continuity, $u \equiv v \equiv f_2$ on $P \cap Q$ and $u \equiv v \equiv f_1$ on $Q \cap R$. Also $p(w)(x) = 0$ for all x; this is clear on $P \cup R$ and on Q it holds because $p(u)(x) \leqslant 0 \leqslant p(v)(x)$ and $p(u)(x) = p(v)(x)$. Finally, w is continuous because it is the combination of continuous functions defined on closed subsets. □

We have given such an extensive proof of Proposition 3.2 because it contains elements that we will use quite often in what follows, to wit breaking the space into closed pieces according to the position of the $f_i(x)$ with respect to $u(x)$ and $v(x)$, and defining w by cases. From now on we assume that $n \geqslant 3$.

To begin, for every x we have $f_n(x) \geqslant u(x) \geqslant f_{n-1}(x)$ or $f_{n-2}(x) \geqslant u(x) \geqslant f_{n-3}(x)$ etc., *because* $p(u)(x) \leqslant 0$; if, for example, $f_{n-1}(x) > u(x) > f_{n-2}(x)$ then clearly $p(u)(x) > 0$. This sequence ends with $f_2(x) \geqslant u(x) \geqslant f_1(x)$ if n is even and with $f_1(x) \geqslant u(x)$ if n is odd.

Likewise, for all x we have $v(x) \geqslant f_n(x)$ or $f_{n-1}(x) \geqslant v(x) \geqslant f_{n-2}(x)$ or $\ldots$ or $f_1(x) \geqslant v(x)$ if n is even and $f_2(x) \geqslant v(x) \geqslant f_1(x)$ if n is odd.

We shall also employ the cover of X by the sets $P = \operatorname{cl}\bigl\{x : u(x) < v(x)\bigr\}$, $Q = \bigl\{x : u(x) = v(x)\bigr\}$ and $R = \operatorname{cl}\bigl\{x : u(x) > v(x)\bigr\}$. On Q there is no choice: the only admissible solution is $w_Q = u \restriction Q = v \restriction Q$. However, once we have found solutions w_P on P and w_R on R then $w = w_P \triangledown w_Q \triangledown w_R$ is the desired solution. On P we have $u \leqslant w_P \leqslant v$ so by continuity we know that $u(x) = w_P(x) = v(x)$ for all $x \in P \cap Q$. Likewise $u(x) = w_R(x) = v(x)$ for all $x \in Q \cap R$. Thus, w is well-defined and as a combination of continuous functions defined on closed subsets it is continuous.

Because hereditary indecomposability is a closed hereditary property we can work inside P and R respectively without worrying about the rest of X.

3.1. **Reduction to odd n.** Assume n is even and recall that in this case $f_1 \leqslant u \leqslant f_n$.

We show that on P we have $q(u) \leqslant 0 \leqslant q(v)$, where $q(t) = \prod_{i=2}^{n}(t - f_i)$. Indeed the possible positions of $u(x)$ ensure that $q(u)(x) \leqslant 0$ for all x. Also, for all x with $u(x) < v(x)$ we have $v(x) \geqslant f_2(x)$ because $f_1(x) \leqslant u(x) < v(x) < f_2(x)$ would imply $p(v)(x) < 0$. Hence, by continuity, $v \geqslant f_2$ on P, so that $q(v) \geqslant \mathbf{0}$ on P.

On the set R we can show in a similar fashion that $v \leqslant f_{n-1}$ and hence that $r(u) \geqslant \mathbf{0} \geqslant r(v)$, where $r(t) = \prod_{i=1}^{n-1}(t - f_i)$.

Both q and r are of degree $n - 1$.

From now on we assume $n \geqslant 3$ and n odd.

3.2. **Reduction to $u \leqslant v$.** If $u(x) > v(x)$ then, because $f_n \geqslant u$, we must have $v(x) \leqslant f_{n-1}(x)$ and because $v \geqslant f_1$ we must have $u(x) \geqslant f_2(x)$. So on R we get, by continuity, $v \leqslant f_{n-1}$ and $u \geqslant f_2$. Consider now the polynomial $q(t) = \prod_{i=2}^{n-1}(t - f_i)$. Because of the possible positions for $u(x)$ and $v(x)$ listed above we conclude that $q(u) \geqslant \mathbf{0} \geqslant q(v)$.

3.3. **The final case.** We now show how to produce w, given that 1) X is the closure of $\bigl\{x : u(x) < v(x)\bigr\}$, 2) $p(u) \leqslant p(v)$ and 3) n is odd.

Let k be such that $n = 2k + 1$. For each $i \leqslant k$ consider the closed sets $A_i = \mathrm{cl}\{x : v(x) < f_{2i+1}(x)\}$ and $B_i = \mathrm{cl}\{x : u(x) > f_{2i-1}(x)\}$.

Note that, because of the positioning of the values $u(x)$ and $v(x)$ we have $A_i \subseteq C_i = \{x : v(x) \leqslant f_{2i}(x)\}$ and $B_i \subseteq D_i = \mathrm{cl}\{x : u(x) \geqslant f_{2i}(x)\}$. Now note that $C_i \cap D_i \subseteq \{x : u(x) = f_{2i}(x) = v(x)\}$; as the set on the right-hand side is nowhere dense it follows that $\mathrm{int}\, C_i$ and $\mathrm{int}\, D_i$ are disjoint.

Also, because X is an F-space, we know that $A_i \subseteq \mathrm{int}\, C_i$ and $B_i \subseteq \mathrm{int}\, D_i$.

Now apply hereditary indecomposability to find three closed sets X_i, Y_i and Z_i that cover X and with the following properties: $A_i \subseteq X_i$, $B_i \subseteq Z_i$, $X_i \cap Y_i \subseteq \mathrm{int}\, D_i$, $Y_i \cap Z_i \subseteq \mathrm{int}\, C_i$ and $X_i \cap Z_i = \varnothing$. We note the following facts:

(1) $u \leqslant f_{2i-1}$ on $X_i \cup Y_i$ because this set is disjoint from B_i;
(2) $v \geqslant f_{2i+1}$ on $Y_i \cup Z_i$ because this set is disjoint from A_i;
(3) $u = f_{2i-1} = f_{2i}$ on $X_i \cap Y_i$ because this set is contained in D_i and because of (1);
(4) $v = f_{2i+1} = f_{2i}$ on $Y_i \cap Z_i$ because the set is contained in C_i and because of (2); and
(5) $u \leqslant f_{2i-1} \leqslant f_{2i} \leqslant f_{2i+1} \leqslant v$ on Y_i because of (1) and (2).

Now we are ready to define w. We start by letting $w = f_1$ on X_1 and $w = f_2$ on Y_1. We continue by letting, for $i > 1$, $w = f_{2i-1}$ on $X_i \cap \bigcap_{j<i} Z_j$ and $w = f_{2i}$ on $Y_i \cap \bigcap_{j<i} Z_j$. Finally, on $\bigcap_{i \leqslant k} Z_i$ we let $w = f_n$.

We check that w is well-defined. By (3) we have $f_{2i-1} = f_{2i}$ on $X_i \cap Y_i$ for every i. If $j < i$ then $X_j \cap X_i \cap \bigcap_{l<i} Z_l = \varnothing$; on $Y_j \cap X_i \cap \bigcap_{l<i} Z_l \subseteq Y_j \cap Z_j \cap Z_{i-1}$ we have $v = f_{2j+1} = f_{2j}$ and $v \geqslant f_{2i-1}$ and so $f_{2i-1} = f_{2j}$, and on $Y_j \cap Y_i \cap \bigcap_{l<i} Z_l \subseteq Y_j \cap Z_j \cap Z_{i-1}$ we have $v = f_{2j=1} = f_{2j}$ and $v \geqslant f_{2i+1} \geqslant f_{2i}$ and so $f_{2j} = f_{2i}$. Finally, on $Y_j \cap \bigcap_{i \leqslant k} Z_i \subseteq Y_j \cap Z_j$ we have $v \geqslant f_n$ and $v = f_{2j}$ so $f_{2j} = f_n$.

We check that $u \leqslant w \leqslant v$. On X_1 we surely have $u \leqslant f_1 \leqslant v$ and if $i > 1$ then on $X_i \cap \bigcap_{j<i} Z_j$ we have $u \leqslant f_{2i-1}$ because of (1) and $f_{2i-1} \leqslant v$ because of (1) for $i - 1$. On each Y_i we have $u \leqslant f_{2i} \leqslant v$ by (5). Finally, on $\bigcap_{i \leqslant k} Z_i$ we have $v \geqslant f_n \geqslant u$ by (2).

We see that w is a well-defined continuous function on X such that $u \leqslant w \leqslant v$ and, because for all x there is an i with $w(x) = f_i(x)$, such that $p(w) = \mathbf{0}$.

4. Questions and Conjectures

The basic question as to what actually characterizes IVT-spaces remains. On the basis of the evidence from Section 3 we conjecture that hereditarily indecomposable spaces also satisfy the IVT for monic polynomials.

The general case seems more complicated in that the leading coefficient (and others) may vanish at certain places. It may very well be that that the full IVT characterizes zero-dimensionality.

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