THE KATOWICE PROBLEM FOR ANALYSTS

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For my long term neighbour

Abstract. The Katowice Problem is well known among topologists and set theorists. The aim of this paper is to make it known among analysts and to give Ben something to think about in his retirement.

Introduction

The Katowice problem, as posed by Marian Turzański, is about Čech-Stone remainders of discrete spaces. For the purposes of this paper it suffices to know that for a discrete space X its Čech-Stone compactification, βX, is a compact Hausdorff space that contains X as a dense subset and that has the property that disjoint subsets of X have disjoint closures in βX. The *remainder* (or *growth*) is $\beta X \setminus X$ and it is generally denoted X^*.

Let X and Y be two infinite sets, endowed with the discrete topology. The problem under consideration asks

The Katowice Problem. If the remainders X^* and Y^* are homeomorphic must there be a bijection between X and Y?

This problem has its origins in Parovichenko's paper [12] where a topological characterization of $\mathbb{N}^*$ is given, under the assumption of the Continuum Hypothesis. The obvious question then is whether such characterizations are possible for the remainders of other discrete spaces, and a natural side question is what can be said if the remainders are homeomorphic.

More information on the Čech-Stone compactification of discrete spaces, and in particular on $\beta\mathbb{N}$ we refer to Van Mill's survey [11].

In this paper we discuss various equivalent versions of the Katowice problem, algebraic and analytic. We also summarise what is known about the problem: the answer is positive except for the case of the first two infinite cardinal numbers. That last remaining case has withstood many attacks thus far and it is hoped that an analytic approach may shed new light on the problem.

1. Another road to the problem

One can arrive at the Katowice Problem by a purely algebraic road. This road starts with an elementary exercise: given a bijection f between two sets X and Y,

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construct a bijection between their power sets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$. The solution is easy: define F by $F(A) = f[A]$.

Now turn this exercise around: given a bijection F between the power sets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ of the sets X and Y, construct a bijection between X and Y.

This second exercise is way more difficult than the first one. The case of finite sets is easily dispensed with it seems: the function $n \mapsto 2^n$ is injective on the set of natural numbers, so if the finite sets X and Y have the same numbers of subsets then X and Y will have the same number of elements. However, this does not solve the problem as required: from the given bijection F construct another bijection. And that last thing cannot be done.

To see this we consider Cohen's original proof that the Continuum Hypothesis is not provable from the axioms of ZFC. In the resulting model there is a bijection between the power sets of the first two infinite cardinal numbers, ω_0 and ω_1, yet there is, of course, no bijection between these sets themselves. This implies that, without using additional properties about the sets in question it is not possible to turn a bijection between the power sets into a bijection between the sets themselves.

I recommend Kunen's book, [10, Chapter VII], for an exposition of Cohen's method.

The situation changes if one considers additional structure. The power set of a set is partially ordered by inclusion and it is even a Boolean algebra, with $\cap$ and $\cup$ as its operations.

Now, from an isomorphism $F : \mathcal{P}(X) \to \mathcal{P}(Y)$ it is quite easy to extract a bijection between X and Y. Indeed, the singleton subsets of X are the atoms of the algebra $\mathcal{P}(X)$; where $a \in \mathcal{P}(X)$ is an atom if $a > \emptyset$ and whenever $a = b \cup c$ one must have $b = a$ or $c = a$. The isomorphism F must then contain a bijection between the respective sets of atoms, which then is a bijection between X and Y of course.

To get to the Katowice problem we ask what happen when we hide the atoms, that is, when we set the atoms equal to zero. In algebraic terms this amounts to taking the ideal, *fin*, of finite sets and considering the quotient algebra $\mathcal{P}(X)/\mathit{fin}$.

The Katowice Problem. If the Boolean algebras $\mathcal{P}(X)/\mathit{fin}$ and $\mathcal{P}(Y)/\mathit{fin}$ are isomorphic is there then a bijection between X and Y?

That this is indeed a reformulation of the Katowice Problem follows readily using M. H. Stone's duality theorem for Boolean algebras. The space βX is the Stone space of the Boolesn algebra $\mathcal{P}(X)$ and the Čech-Stone remainder X^* is the Stone space of $\mathcal{P}(X)/\mathit{fin}$.

More information about Stone's duality theorem can be found in Koppelberg's book, [8, Chapter 3].

2. Rings, and Banach algebras and lattices

Yet another way of looking at the Katowice problem is via the function space ℓ_∞. That is, for every set X we consider $\ell_\infty(X)$, the set of all bounded real- (or complex-)valued functions on the set X.

One can consider $\ell_\infty(X)$ as a ring, by defining addition and multiplication pointwise, and as a Banach-algebra, by endowing it with the supremum norm $\|\cdot\|_\infty$, and, most importantly, as a Banach lattice under the pointwise order.

In all three cases, if $\ell_\infty(X)$ and $\ell_\infty(Y)$ are isomorphic then there is a bijection between X and Y. This follows by applying the theorem of Gel′fand and Kolmogorov ([5]) in the case of rings, or that of Gel′fand and Neumark ([6]) in the case of Banach-algebras, or that of Kakutani ([7]) and Krein and Krein ([9]) in the case of Banach lattices.

These theorems represent $\ell_\infty(X)$ as the ring, or Banach-algebra, or Banach lattice of continuous functions, respectively, on a certain compact Hausdorff space. In this case that space is just βX, the Čech-Stone compactification of the discrete space X.

Just as in the case of Boolean algebras one can hide the finite sets by taking the quotient $\ell_\infty(X)/c_0$ by the ideal or subalgebra or ideal, respectively, of functions that vanish at infinity, where $f : X \to \mathbb{R}$ *vanishes at infinity* if for every $\varepsilon > 0$ the set $\{x : |f(x)| \geqslant \varepsilon\}$ is finite.

The quotient $\ell_\infty(X)/c_0$ corresponds to the ring, or Banach-algebra, or Banach lattice of continuous functions on the Čech-Stone remainder X^* and thus we come to a version reformulation of the Katowice problem that, I hope, is of interest to analysts.

The Katowice Problem. If the Banach lattices $\ell_\infty(X)/c_0$ and $\ell_\infty(Y)/c_0$ are isomorphic is there then a bijection between X and Y?

3. What is known?

To begin: the Generalised Continuum Hypothesis (GCH) implies that the Katowice Problem has a positive answer. The Boolean algebraic version makes this clear: the Boolean algebra $\mathcal{P}(X)/\mathit{fin}$ has cardinality $2^{|X|}$, and the GCH implies that the function $\kappa \mapsto 2^\kappa$ is injective on the class of cardinal numbers.

In fact, much more is known. In joint work Balcar and Frankiewicz established that the answer is actually positive without any additional set-theoretic assumptions, *when the two sets are both uncountable.* More precisely

Theorem ([1,4])**.** *If the remainders X^* and Y^* are homeomorphic and the sets X and Y are uncountable then there is a bijection between X and Y.*

In fact, this theorem leaves just one pair of cardinal numbers for which the problem is still open: the first two infinite cardinals numbers ω_0 and ω_1. I use 'cardinal numbers' rather than 'sets' because it would become increasingly cumbersome to formulate everything in terms of arbitrary sets.

The cardinal numbers form a class of well-ordered sets against which all other sets are measured: for every set X there is one cardinal number κ such that there is a bijection between X and κ. We write $\kappa = |X|$ and call κ the cardinal number of X. By using the word 'one' we implicitly specify that there are no bijections between distinct cardinal numbers.

All this reduces the Katowice Problem to one final case.

Main Problem. Prove that $\ell_\infty(\omega_0)/c_0$ and $\ell_\infty(\omega_1)/c_0$ are not isomorphic.

This formulation reflects this author's preferred solution of this problem. I am fully aware that it is relatively consistent with ZFC that $\ell_\infty(\omega_0)/c_0$ and $\ell_\infty(\omega_1)/c_0$ are isomorphic. However, to me that would be too shocking to be true.

In the next section I will list some consequences derived from the assumption that the two lattices are isomorphic. The reader will see that this has almost developed into a game where someone derives a consequence and someone else shows that that consequence does not lead to a contradiction, not even when combined with earlier consequences.

4. Some consequences

Most of the consequences have been obtained in the Boolean algebraic setting so we adopt that language from now on. Thus, our standing assumption is that there is an isomorphism $\gamma : \mathcal{P}(\omega_0)/\mathit{fin} \to \mathcal{P}(\omega_1)/\mathit{fin}$.

The first consequence of this is straightforward: the cardinalities of the Boolean algebras are the same, so we obtain

Consequence 1. $2^{\aleph_0} = 2^{\aleph_1}$.

We have already seen that this consequences does not lead to a contradiction.

To describe the other consequences we introduce some notation. First we change the underlying sets to $\mathbb{Z} \times \omega_0$ and $\mathbb{Z} \times \omega_1$, where $\mathbb{Z}$ is the set of integers.

In the product $\mathbb{Z} \times \omega_1$ we distinguish a few special sets:

(1) $V_n = \{n\} \times \omega_1$ $(n \in \mathbb{Z})$, the vertical lines
(2) $H_\alpha = \mathbb{Z} \times \{\alpha\}$ $(\alpha \in \omega_1)$, the horizontal lines
(3) $E_\alpha = \mathbb{Z} \times [\alpha, \omega_1)$ $(\alpha \in \omega_1)$, the end segments, here $[\alpha, \omega_1)$ is a convenient short hand for the set $\{\beta \in \omega_1 : \beta \geqslant \alpha\}$

If A is a subset of $\mathbb{Z} \times \omega_0$ or $\mathbb{Z} \times \omega_1$ then A^* denotes its equivalence class modulo *fin*.

Back in $\mathcal{P}(\mathbb{Z} \times \omega_0)$ we choose sets v_n, h_α and e_α such that $\gamma(v_n^*) = V_n^*$, $\gamma(h_\alpha^*) = H_\alpha^*$, and $\gamma(e_\alpha^*) = E_\alpha^*$. The relations between the sets V_n, H_α and E_α are mirrored by those between the sets v_n, h_α and e_α. For example $H_\alpha \cap H_\beta = \emptyset$ if $\alpha < \beta$, so in $\mathcal{P}(\mathbb{Z} \times \omega_0)/\mathit{fin}$ we have $h_\alpha^* \wedge h_\beta^* = 0$; for the sets themselves this means that $h_\alpha \cap h_\beta$ belongs to *fin*. We write this as $h_\alpha \cap h_\beta =^* \emptyset$ and say that h_α and h_β are *almost disjoint*.

Likewise when $\alpha < \beta$ we have $E_\beta \subseteq E_\alpha$ and hence $e_\beta^* \leqslant e_\alpha^*$, which means that $e_\beta \setminus e_\alpha$ is finite; we abbreviate the latter by $e_\beta \subseteq^* e_\alpha$. In fact $E_\alpha \setminus E_\beta$ is finite, hence so is $e_\alpha \setminus e_\beta$; we should therefore actually write $e_\beta \subset^* e_\alpha$.

The sets v_n are also almost disjoint but we may alter each of them by a finite set (and apply a bijection from $\mathbb{Z} \times \omega_0$ to itself) so that we achieve that $v_n = \{n\} \times \omega_0$ for all n.

One of the (admittedly feeble) reasons for believing that $\mathcal{P}(\omega_0)/\mathit{fin}$ and $\mathcal{P}(\omega_1)/\mathit{fin}$ are *not* isomorphic is the shape of the two products: $\mathbb{Z} \times \omega_0$ looks like a squat 2×1-rectangle, while $\mathbb{Z} \times \omega_1$ is a rectangle with the same base that is much taller than the first product. It seems inconceivable we can squeeze the uncountable stack of H_α's into the flat rectangle that is $\mathbb{Z} \times \omega_0$.

We turn to a less trivial consequence of having our supposed isomorphism γ.

Consider the end segments E_α and their companion sets e_α. Since $V_n \cap E_\alpha$ is always uncountable, the intersection $v_n \cap e_\alpha$ is always infinite. This means that we can define $f_\alpha : \mathbb{Z} \to \omega_0$ by

$$f_\alpha(n) = \min\{m : \langle n, m \rangle \in e_\alpha\}$$

The resulting sequence $\langle f_\alpha : \alpha \in \omega_1 \rangle$ of functions has two interesting properties.

We just saw that $e_\beta \setminus e_\alpha$ is finite and $e_\alpha \setminus e_\beta$ is infinite when $\alpha < \beta$. From this we get the first property: $f_\alpha(n) \leqslant f_\beta(n)$ for all but finitely many n. This is generally abbreviated as $f_\alpha \leqslant^* f_\beta$.

For the second let $f : \mathbb{Z} \to \omega_1$ be arbitrary and consider the set

$$l_f = \{\langle n, m\rangle : m \leqslant f(n)\}.$$

In $\mathcal{P}(\mathbb{Z} \times \omega_1)$ choose a set L_f such that $\gamma(l_f^*) = L_f^*$.

For every n the intersection $v_n \cap l_f$ is finite, hence so is $V_n \cap L_f$. This implies that there is an α such that $E_\alpha \cap L_f = \emptyset$. Back at the ω_0-side we find that $e_\alpha \cap l_f$ is finite. But this then implies that $f(n) < f_\alpha(n)$ for all but finitely many n.

We see that $\langle f_\alpha : \alpha \in \omega_1\rangle$ is both increasing and cofinal with respect to the (quasi-)order $\leqslant^*$. Such a sequence is called an ω_1-scale.

Consequence 2. *There is an ω_1-scale.*

The existence of an ω_1-scale follows from the Continuum Hypothesis (CH) but it is also consistent with the latter's negation and, specifically, also with $2^{\aleph_0} = 2^{\aleph_1}$.

The next consequence involves the horizontal lines H_α and their counterparts the h_α. We have already seen that the h_α form an *almost disjoint family*; we now show that it is a very special such family.

Suppose that for every α we choose a subset x_α of h_α. At the side of ω_1 we choose $X_\alpha \subseteq H_\alpha$. such that $\gamma(x_\alpha^*) = X_\alpha^*$ and we take the union $X = \bigcup_{\alpha\in\omega_1} X_\alpha$. Then we know that $X \cap H_\alpha = X_\alpha$ for all α. If we then choose x with $\gamma(x^*) = X^*$ then this x will satisfy $x \cap h_\alpha =^* x_\alpha$ for all α, where $=^*$ means 'almost equal' (again: but for finitely many points).

We say that $\{h_\alpha : \alpha < \omega_1\}$ is a *uniformizable* almost disjoint family.

Consequence 3. *There is a uniformizable almost disjoint family of cardinality $\aleph_1$ (also called a strong Q-sequence).*

The existence of a uniformizable almost disjoint family implies the equality $2^{\aleph_0} = 2^{\aleph_1}$: to code a subset Y of ω_1 apply the previous paragraph with $x_\alpha = h_\alpha$ if $\alpha \in Y$ and $x_\alpha = \emptyset$ if $\alpha \notin Y$ to obtain x_Y. The map $Y \mapsto x_Y$ is injective from $\mathcal{P}(\mathbb{Z} \times \omega_1)$ into $\mathcal{P}(\mathbb{Z} \times \omega_0)$.

That the assumptions $2^{\aleph_0} = 2^{\aleph_1}$ and 'there is an ω_1-scale' do not together lead to $0 = 1$ is an easy exercise for anyone who has learned the rudiments of forcing (use the random real model). To show the same thing for the combination of 'there is a strong Q-sequence' and 'there is an ω_1-scale' is not such an easy exercise. But it can be done, see [2] for a proof.

We treat one more consequence, which involves automorphisms of the Boolean algebras. An algebra like $\mathcal{P}(X)/\mathit{fin}$ has many automorphisms: every permutation of X and, more generally, every bijection between co-finite subsets of X determines an automorphism of $\mathcal{P}(X)/\mathit{fin}$. Such automorphisms are called trivial. It is a remarkable result of Shelah's ([13, Chapter IV]) that it is consistent that all automorphisms of $\mathcal{P}(\omega_0)/\mathit{fin}$ are trivial. This was extended to *all* sets by Veličković in [14].

Using our (postulated) isomorphism γ we can show that both $\mathcal{P}(\omega_0)$ and $\mathcal{P}(\omega_1)$ must have non-trivial automorphisms. We describe these automorphisms and refer to [3] for proofs.

For the first take the shift on ω_0 (which is set theory's set of natural numbers): $\sigma(n) = n+1$. The automorphism that we get by transplanting the automorphism $A^* \mapsto \sigma[A]^*$ to $\mathcal{P}(\omega_1)/\mathit{fin}$ is non-trivial.

For the second take the map $\tau : \mathbb{Z} \times \omega_1 \to \mathbb{Z} \times \omega_1$ given by $\tau(n, \alpha) = \langle n+1, \alpha\rangle$. Transplanting $A^* \mapsto \tau[A]^*$ to $\mathcal{P}(\mathbb{Z}\times\omega_0)/\mathit{fin}$ is results in a non-trivial automorphism.

Consequence 4. *Both $\mathcal{P}(\omega_0)/\mathit{fin}$ and $\mathcal{P}(\omega_0)/\mathit{fin}$ have non-trivial automorphisms.*

The paper [3] contains some more consequences and references to other sources of information about the Katowice problem

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