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A CONCRETE CO-EXISTENTIAL MAP THAT IS NOT CONFLUENT

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Abstract. We give a concrete example of a co-existential map between continua that is not confluent.

Introduction

In [1], Paul Bankston gives an example of a co-existential map that is not confluent. The construction is rather involved and does not produce a concrete example of such a map. A lot of effort is needed to get the main ingredient, to wit, a co-diagonal map that is not monotone.

The purpose of this note is to show that one can write down a concrete map between two rather simple continua that is co-existential and not confluent. It will be clear from the construction that the range space admits co-diagonal maps that are nor confluent and, a fortiori, not monotone.

1. Preliminaries

In the interest of brevity, we try to keep the notation down to the bare minimum.

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2.1 Ultra-copowers and associated maps

Given a compact space Y and a set I, we consider the Čech-Stone compactification $\beta(Y \times I)$, where I carries the discrete topology. There are two useful maps associated with $\beta(Y \times I)$: the Čech-Stone extensions of the projections $\pi_Y : Y \times I \to Y$ and $\pi_I : Y \times I \to I$. Given an ultrafilter u on I, we write $Y_u = \beta\pi_I^{\leftarrow}(u)$ and we let $q_u = \beta\pi_Y \restriction Y_u$. In the terminology of [1], the space Y_u is the *ultra-copower* of Y by the ultrafilter u and $q_u : Y_u \to Y$ is the associated *co-diagonal map*. A map $f : X \to Y$ between compact spaces is *co-existential* if there are a set I, an ultrafilter u on I, and a map $g : Y_u \to X$ such that $q_u = f \circ g$.

These notions can be seen as dualizations of notions from model theory and they offer inroads to the study of compact Hausdorff spaces by algebraic and, in particular, lattice-theoretic means.

2.2 Two notions from continuum theory

On a first-order algebraic level there is not much difference between Y and Y_u: they have elementarily equivalent lattice-bases for their closed sets; the map $A \mapsto Y_u \cap \mathrm{cl}_\beta(A \times I)$ is an elementary embedding of such bases. It is, therefore, not unreasonable to expect that the co-diagonal map q_u be well-behaved. For example, one could expect it to be *confluent*, which means that if C is a subcontinuum of Y then every component of $q_u^{\leftarrow}[C]$ would be mapped onto C by q_u. Certainly *some* component of $q_u^{\leftarrow}[C]$ is mapped onto C: the component that contains $Y_u \cap \mathrm{cl}_\beta(C \times I)$ (this shows that q_u is *weakly* confluent). Intuitively, there should be no difference between the components, so all should be mapped onto C. The example below disproves this intuition.

In [1], Bankston gives (references for) other reasons why it is of interest to know whether co-diagonal and co-existential maps are confluent.

2. The example

We start with the closed infinite broom [3, 120, p. 139]

$$B = \bigl([0,1] \times \{0\}\bigr) \cup \bigcup_{n\in\omega} H_n$$

where $H_n = \{\langle t, t/2^n\rangle : 0 \le t \le 1\}$ is the nth hair of the broom.

The range space is B with the limit hair extended to have length 2:

$$Y = B \cup \bigl([1,2] \times \{0\}\bigr).$$

We denote the extended hair $[0,2] \times \{0\}$ by H_ω.

The domain of the map is B with an extra hair of length 2 along the y-axis:

$$X = B \cup \bigl(\{0\} \times [0,2]\bigr).$$

The map $f : X \to Y$ is the (more-or-less) obvious one:

$$f(x,y) = \begin{cases} \langle x,y \rangle & \langle x,y \rangle \in B \\ \langle y,0 \rangle & x = 0\,. \end{cases}$$

Thus, f is the identity on B and it rotates the points on the extra hair over $-\frac{1}{2}\pi$.

Claim 1. *The map f is not confluent.*

Proof: This is easy. The components of the preimage of the continuum $C = [1,2] \times \{0\}$ are the interval $\{0\} \times [1,2]$ and the singleton $\{\langle 1,0 \rangle\}$; the latter does not map onto C. $\square$

Claim 2. *The map f is co-existential.*

Proof: We need to find an ultrafilter u and a map $g : Y_u \to X$ such that $f \circ g$ is the co-diagonal map $q_u : Y_u \to Y$. In fact, any free ultrafilter u on ω will do.

We define two closed subsets F and G of $Y \times \omega$ and define g on the intersections $F_u = Y_u \cap \mathrm{cl}_\beta F$ and $G_u = Y_u \cap \mathrm{cl}_\beta G$ separately. We set

$$F = \bigcup_{n \in \omega} \biggl(\bigcup_{k \le n} \bigl(H_k \times \{n\}\bigr) \biggr)$$

and

$$G = \bigcup_{n \in \omega} \biggl(\bigcup_{n < k \le \omega} \bigl(H_k \times \{n\}\bigr) \biggr).$$

Note that $F \cup G = Y \times \omega$ and that $F \cap G = \{\langle 0,0 \rangle\} \times \omega$, so that $F_u \cup G_u = Y_u$ and $F_u \cap G_u$ consists of one point, the (only) accumulation point of $F \cap G$ in Y_u.

It is an elementary verification that $q_u[F_u] = B$ and $q_u[G_u] = H_\omega$. This allows us to define $g : Y_u \to X$ by cases: on F_u, we define g to be just q_u, and on G_u, we define $g = R \circ q_u$, where R rotates the plane over $\frac{1}{2}\pi$. These definitions agree at the point in $F_u \cap G_u$ and give continuous maps on F_u and G_u, respectively. Therefore, the combined map $g : Y_u \to X$ is continuous as well. □

This also shows that the co-diagonal map q_u is not confluent; no component of the preimage under g of $\langle 1, 0\rangle$ is mapped onto C.

Remark. In [2], Bankston shows that if a continuum K is such that every co-existential map onto K is confluent, then every K must be connected im kleinen at each of its cut points. The continuum Y above is connected im kleinen at all cut points but one: the point $\langle 1, 0\rangle$. So Y does not qualify as a counterexample to the converse.

To obtain a counterexample, multiply X and Y by the unit interval and multiply f by the identity. The proof that the new map is co-existential but not confluent is an easy adaptation of the proof that f has these properties. Since Y has no cut points, it is connected im kleinen at all of them.

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