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Applications of another characterization of $\beta\mathbb{N}\setminus\mathbb{N}$

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Abstract

Steprāns provided a characterization of $\beta\mathbb{N}\setminus\mathbb{N}$ in the $\aleph_2$-Cohen model that is much in the spirit of Parovičenko's characterization of this space under CH. A variety of the topological results established in the Cohen model can be deduced directly from the properties of $\beta\mathbb{N}\setminus\mathbb{N}$ or $\mathcal{P}(\mathbb{N})/\mathit{fin}$ that feature in Steprāns' result. © 2002 Elsevier Science B.V. All rights reserved.

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Introduction

Topological problems that involve the behaviour of families of subsets of the set of natural numbers tend to have (moderately) easy solutions if the Continuum Hypothesis (CH) is assumed. The reason for this is that one's inductions and recursions last only $\aleph_1$ steps and that at each intermediate step only countably many previous objects have to be dealt with.

An archetypal example is Parovičenko's characterization, see [22], of the space $\mathbb{N}^*$ as the only compact zero-dimensional F-space of weight $\mathfrak{c}$ without isolated points in which non-empty G_δ-sets have non-empty interiors. The proof actually shows that $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is the unique atomless Boolean algebra of size $\mathfrak{c}$ with a certain property R_ω and then applies Stone duality to establish uniqueness of $\mathbb{N}^*$. It runs as follows: consider two Boolean

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algebras A and B with the properties of $\mathcal{P}(\mathbb{N})/\mathit{fin}$ just mentioned and well-order both in type ω_1. Assume we have an isomorphism h between subalgebras A_α and B_α that contain $\{a_\beta\colon \beta < \alpha\}$ and $\{b_\beta\colon \beta < \alpha\}$ respectively. We need to define $h(a_\alpha)$ (if $a_\alpha \notin A_\alpha$); consider

$$S = \{a \in A_\alpha\colon a < a_\alpha\} \quad \text{and} \quad T = \left\{a \in A_\alpha\colon a < a'_\alpha\right\}.$$

We must find $b \in B$ such that

(1) $h(a) < b$ if $a \in S$;
(2) $h(a) < b'$ if $a \in T$;
(3) $h(a) \wedge b \neq 0$ and $h(a) \wedge b' \neq 0$ if $a \in A_\alpha$ but $a \notin S \cup T$.

Property R_ω says exactly that this is possible—its proper formulation can be found after Definition 2.9.

This paper grew out of observations that in the Cohen model the Boolean algebra $\mathcal{P}(\mathbb{N})/\mathit{fin}$ retains much of the properties that were used above. In a sense to be made precise later, in Definition 2.3, $\mathcal{P}(\mathbb{N})/\mathit{fin}$ contains many subalgebras that are like A_α and B_α above ($\aleph_0$-ideal subalgebras); even though these will not be countable the important sets S and T will be. We also define a cardinal invariant, $\mathfrak{m}_c$, that captures just enough of R_ω to allow a Parovičenko-like characterization of $\mathcal{P}(\mathbb{N})/\mathit{fin}$ in the $\aleph_2$-Cohen model—this is Steprāns' result alluded to in the abstract (Theorem 2.13). During the preparation of this paper we became aware of recent work on the weak Freese–Nation property in [14,15,13]. Although the weak Freese–Nation property is stronger than our properties the proofs of the consequences are very similar; therefore we restrict, with few exceptions, ourselves to more topological (and new) applications. Perhaps the difference in approach (weak Freese–Nation versus $(\aleph_1, \aleph_0)$-ideal) is mostly a matter of taste but ours arose directly out of Steprāns original results and the essentially folklore facts about the effects of adding Cohen reals.

In Section 2 we shall formulate the properties alluded to above and prove that in the Cohen model $\mathcal{P}(\mathbb{N})/\mathit{fin}$ does indeed satisfy them. In Section 3 we select some results about $\mathcal{P}(\mathbb{N})/\mathit{fin}$ (or $\mathbb{N}^*$) that are known to hold in the Cohen model and derive them directly from the new properties—whenever we credit a result to some author(s) we mean to credit them with establishing that it holds in the Cohen model. In Sections 4 and 5 we investigate the properties themselves and their behaviour with respect to subalgebras and quotients. Finally, in Section 6 we investigate how much of an important phenomenon regarding $\mathbb{N}^*$ persists; we are referring to the fact that under CH for every compact zero-dimensional space X of weight $\mathfrak{c}$ or less the Čech–Stone remainder $(\omega \times X)^*$ is homeomorphic to $\mathbb{N}^*$.

We would like to take this opportunity to thank the referee for a very insightful remark concerning our version of Bell's example (see Definition 3.7), which enabled us to simplify the presentation considerably.

1. Preliminaries

Boolean algebras. Our notation is fairly standard: b' invariably denotes the complement of b.

For a subset S of a Boolean algebra B, let $S^\perp$ denote the ideal of members of B that are disjoint from every element of S, i.e., $S^\perp = \{b \in B\colon (\forall s \in S)(b \wedge s = 0)\}$. For convenience, we use $b^{\perp}$ in place of $\{b\}^{\perp}$. Also, let $b^{\downarrow}$ be the principal ideal generated by b, namely $\{\alpha \in B\colon a \leqslant b\}$. Clearly $b^{\downarrow}$ is equal to $(b')^{\perp}$. Also, for subsets S and T we let $S \perp T$ abbreviate $(\forall s \in S)(\forall t \in T)(s \wedge t = 0)$; in fact we shall often abbreviate $s \wedge t = 0$ by $s \perp t$.

Cohen reals. 'The Cohen model' is any model obtained from a model of the GCH by adding a substantial quantity of Cohen reals—more than $\aleph_1$. In particular 'the $\aleph_2$-Cohen model' is obtained by adding $\aleph_2$ many Cohen reals. Actually, since we are intent on proving our results using the *properties* of $\mathcal{P}(\mathbb{N})/\mathit{fin}$ only, many readers may elect to take Lemma 2.2, Theorem 2.7 and the remark made after Proposition 2.12 on faith or else consult [18] for the necessary background on Cohen forcing.

The weak Freese–Nation property. A partially ordered set P is said to have the *weak Freese–Nation property* if there is a function $F : P \to [P]^{\aleph_0}$ such that whenever $p \leqslant q$ there is $r \in F(p) \cap F(q)$ with $p \leqslant r \leqslant q$.

Elementary substructures. Consider two structures M and N (groups, fields, Boolean algebras, models of set theory ...), where M is a substructure of N. We say that M is an *elementary* substructure of N, and we write $M \prec N$, if every equation, involving the relations and operations of the structures and constants from M, that has a solution in N has a solution in M as well.

The Löwenheim–Skolem theorem says that every subset A of a structure N can be enlarged to an elementary substructure M of whose cardinality is the maximum of $|A|$ and $\aleph_0$. The construction proceeds in the obvious way: in a recursion of length ω one keeps adding solutions to equations that involve ever more constants.

We prefer to think of an argument that uses elementary substructures as the lazy man's closing off argument; rather than setting up an impressive recursive construction we say "let θ be a suitably large cardinal and let M be an elementary substructure of $H(\theta)$" and add some words that specify what M should certainly contain.

The point is that the impressive recursion is carried out inside $H(\theta)$, where θ is 'suitably large' (most of the time $\theta = \mathfrak{c}^{+}$ is a good choice as everything under consideration has cardinality at most $\mathfrak{c}$), and that it (or a nonessential variation) is automatically subsumed when one constructs an elementary substructure of $H(\theta)$.

In this paper we shall be working mostly with $\aleph_1$-sized elementary substructures, most of which will be $\aleph_0$-*covering*. The latter means that every countable subset A of M is a subset of a countable element B of M. This is not an unreasonable property, considering that the ordinal ω_1 has it: every countable subset of ω_1 is a subset of a countable ordinal.

An $\aleph_0$-covering structure can be constructed in a straightforward way. One recursively constructs a chain $\langle M_\alpha\colon \alpha < \omega_1\rangle$ of countable elementary substructures of $H(\theta)$ with the property that $\langle M_\beta\colon \beta \leqslant \alpha\rangle \in M_{\alpha+1}$ for all α. In the end $M = \bigcup_{\alpha<\omega_1} M_\alpha$ is as required: if $A \subseteq M$ is countable then $A \subseteq M_\alpha$ for some α and $M_\alpha \in M$.

For just a few of the results we indicate two proofs: a direct one and one via elementarity—we invite the reader to compare the two approaches and to reflect on their efficacy.

2. Two new properties of $\mathcal{P}(\mathbb{N})/\mathit{fin}$

In this section we introduce two properties that Boolean algebras may have. We shall prove that in the Cohen model $\mathcal{P}(\mathbb{N})/\mathit{fin}$ has both and that in the $\aleph_2$-Cohen model their conjunction actually characterizes $\mathcal{P}(\mathbb{N})/\mathit{fin}$.

$(\aleph_1, \aleph_0)$-ideal algebras. We begin by defining the $\aleph_0$-ideal subalgebras alluded to in the introduction.

Definition 2.1. For a Boolean algebra B, we will say that a subalgebra A of B is $\aleph_0$-ideal if for each $b \in B \setminus A$ the ideal $\{a \in A\colon a < b\} = A \cap b^{\downarrow}$ has a countable cofinal subset.

Of course, by duality, the ideal $b^{\perp} \cap A$ is countably generated as well; thus in $\aleph_0$-ideal subalgebras the phrase "S and T are countable" from the introduction is replaced by "S and T have countable cofinal subsets".

The main impetus for this definition comes from following result.

Lemma 2.2 [23, Lemma 2.2]. *If G is $\mathrm{Fn}(I,2)$-generic over V then $\mathcal{P}(\mathbb{N}) \cap V$ is an $\aleph_0$-ideal subalgebra of $\mathcal{P}(\mathbb{N})$ in $V[G]$.*

Proof. Let $\widetilde{X}$ be an $\mathrm{Fn}(I,2)$-name for a subset of $\mathbb{N}$. It is a well-known fact about $\mathrm{Fn}(I,2)$ that there is a countable subset J of I such that $\widetilde{X}$ is completely determined by $\mathrm{Fn}(J,2)$. This means that for every $p \in \mathrm{Fn}(I,2)$ and every $n \in \mathbb{N}$ we have $p \Vdash n \in \widetilde{X}$ (or $p \Vdash n \notin \widetilde{X}$) if and only if $p \restriction J$ does.

For every $p \in \mathrm{Fn}(J,2)$ define $X_p = \{n\colon p \Vdash n \in \widetilde{X}\}$; the countable family of these X_p is as required. $\square$

The factoring lemma for Cohen forcing [18, p. 255] implies that for every subset J of I the subalgebra $A_J = \mathcal{P}(\mathbb{N}) \cap V[G \restriction J]$ is $\aleph_0$-ideal in the final $\mathcal{P}(\mathbb{N})$. Using the fact, seen in the proof above, that names for subsets of $\mathbb{N}$ are essentially countable one can verify that $A_{\bigcup \mathcal{J}} = \bigcup_{J \in \mathcal{J}} A_J$ for every chain $\mathcal{J}$ of subsets of I of uncountable cofinality. This shows that in the Cohen model $\mathcal{P}(\mathbb{N})$ has many $\aleph_0$-ideal subalgebras and also that the family of these subalgebras is closed under unions of chains of uncountable cofinality.

What we call $\aleph_0$-ideal is called 'good' in [23] and in [14] the term σ-subalgebra is used. In the latter paper it is also shown that if $F : B \to [B]^{\aleph_0}$ witnesses the weak Freese–Nation property of B then every subalgebra that is closed under F is an $\aleph_0$-ideal subalgebra; therefore an algebra with the weak Freese–Nation property has many $\aleph_0$-ideal subalgebras and the family of these subalgebras is closed under directed unions.

We are naturally interested in Boolean algebras with many $\aleph_0$-ideal subalgebras. Most of our results only require that there are many $\aleph_1$-sized $\aleph_0$-ideal subalgebras.

Definition 2.3. We will say that a Boolean algebra B is $(\aleph_1, \aleph_0)$-ideal if the set of $\aleph_1$-sized $\aleph_0$-ideal subalgebras of B contains an $\aleph_1$-cub of $[B]^{\aleph_1}$. That is, there is a family $\mathcal{A}$ consisting of $\aleph_1$-sized $\aleph_0$-ideal subalgebras of B such that every subset of size $\aleph_1$ is contained in some member of $\mathcal{A}$ and the union of each chain from $\mathcal{A}$ of cofinality ω_1 is again in $\mathcal{A}$.

We leave to the reader the verification that $\mathcal{P}(\mathbb{N})$ is an $(\aleph_1, \aleph_0)$-ideal algebra if and only if $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is an $(\aleph_1, \aleph_0)$-ideal algebra (but see Corollary 5.12). It is also worth noting that $\mathcal{P}(\omega_1)$ is not $(\aleph_1, \aleph_0)$-ideal (see [14, Proposition 5.3]).

Since the definition of $(\aleph_1, \aleph_0)$-ideal requires that we have some $\aleph_1$-cub consisting of $\aleph_0$-ideal subalgebras, it is a relatively standard fact that every $\aleph_0$-covering elementary substructure of size $\aleph_1$ of a suitable $H(\theta)$ induces an $\aleph_0$-ideal subalgebra. We shall use the following lemma throughout this paper, not always mentioning it explicitly—it is an instance of the rule-of-thumb that says: if $X, \mathcal{A} \in M$, where M is suitably closed and $\mathcal{A}$ some sort of cub in $\mathcal{P}(X)$, then $X \cap M \in \mathcal{A}$.

Lemma 2.4. *Let B be an $(\aleph_1, \aleph_0)$-ideal algebra, let θ be a suitably large cardinal and let M be an $\aleph_0$-covering elementary substructure of size $\aleph_1$ of $H(\theta)$ that contains B. Then $B \cap M$ is an $\aleph_0$-ideal subalgebra of B.*

Proof. Note first that M contains an $\aleph_1$-cub $\mathcal{A}$ as in Definition 2.3: it must contain a solution to the equation

$$x \text{ is an } \aleph_1\text{-cub in } [B]^{\aleph_1} \text{ that consists of } \aleph_0\text{-ideal subalgebras.}$$

Let $f : \omega_1 \to \mathcal{A} \cap M$ be a surjection, not necessarily from M. Because M is $\aleph_0$-covering we can find, for every $\alpha \in \omega_1$, a countable element X_α of M that contains $f[\alpha]$. Consider the equation

$$x \in \mathcal{A} \quad \text{and} \quad \bigcup (X_\alpha \cap \mathcal{A}) \subseteq x.$$

This equation has a solution in $H(\theta)$ and hence in M; we may take $A_\alpha \in \mathcal{A} \cap M$ such that $\bigcup (X_\alpha \cap \mathcal{A}) \subseteq A_\alpha$. Thus we construct an increasing chain $\langle A_\alpha \colon \alpha < \omega_1 \rangle$ in $\mathcal{A} \cap M$ that is cofinal in $\mathcal{A} \cap M$. It follows that $\bigcup (\mathcal{A} \cap M) = \bigcup_{\alpha < \omega_1} A_\alpha$ belongs to $\mathcal{A}$. Now check carefully that $B \cap M = \bigcup (\mathcal{A} \cap M)$—use that $\mathcal{A}$ is unbounded in $[B]^{\aleph_1}$. $\square$

The remarks preceding Definition 2.3 show that an algebra with the weak Freese–Nation property is $(\aleph_1, \aleph_0)$-ideal. The converse is almost true—the difference is that we do not require closure under countable unions. In the notation used after Lemma 2.2 the family $\mathcal{A} = \{A_J \colon J \in [I]^{\aleph_1}\}$ witnesses that in the Cohen model $\mathcal{P}(\mathbb{N})$ is always $(\aleph_1, \aleph_0)$-ideal. However $\mathcal{A}$ is *not* closed under unions of countable chains. Indeed, in [12] one finds the theorem that if V satisfies the GCH and the instance $(\aleph_{\omega+1}, \aleph_\omega) \to (\aleph_1, \aleph_0)$ of Chang's conjecture then after adding one dominating real d and then $\aleph_\omega$ Cohen reals $\mathcal{P}(\mathbb{N})$ does not have the weak Freese–Nation property—as $V[d]$ still satisfies the GCH the final model is a 'Cohen model'.

Many properties of $\mathcal{P}(\mathbb{N})$ that hold in the Cohen model can be derived from the weak Freese–Nation property—see [15,13], for example—and many of these can be derived from the fact that $\mathcal{P}(\mathbb{N})$ is $(\aleph_1, \aleph_0)$-ideal. It is not our intention to duplicate the effort of [13]; we will concentrate on topological applications. However, to give the flavour, and because we shall use the result a few times, we consider Kunen's theorem from [17] that in the Cohen model the Boolean algebra $\mathcal{P}(\mathbb{N})/\mathit{fin}$ does not have a chain of order type ω_2.

It is quite straightforward to show that an algebra with the weak Freese–Nation property does not have any well-ordered chains of order type ω_2; with a bit more effort the same can be said of $(\aleph_1, \aleph_0)$-ideal algebras.

Proposition 2.5. *An $(\aleph_1, \aleph_0)$-ideal Boolean algebra does not have any chains of order type ω_2.*

Proof. Assume that $\{c_\alpha\colon \alpha < \omega_2\}$ is an increasing chain in B and let $\mathcal{A}$ be as in Definition 2.3. Recursively construct a chain $\{A_\alpha\colon \alpha \in \omega_1\}$ in $\mathcal{A}$ and an increasing sequence $\{\gamma_\alpha\colon \alpha \in \omega_1\}$ of ordinals in ω_2 as follows. Let $\gamma_0 = 0$ and, given $\mathcal{A}_\beta$ and γ_β for $\beta < \alpha$, let $A = \bigcup_{\beta<\alpha} A_\alpha$ and $\gamma = \sup_{\beta<\alpha} \gamma_\beta$. Choose $A_\alpha \in \mathcal{A}$ such that $c_\gamma \in A_\alpha$ and such that for every $a \in A$, *if* there is a β with $a \leqslant c_\beta$ *then* there is a β such that $a \leqslant c_\beta$ *and* $c_\beta \in A_\alpha$; let γ_α be the first γ for which $c_\gamma \notin A_\alpha$.

In the end set $A = \bigcup\{A_\alpha\colon \alpha \in \omega_1\}$ and $\lambda = \sup\{\gamma_\alpha\colon \alpha \in \omega_1\}$. Now we have a contradiction because although $c_\lambda^\downarrow \cap A$ should be countably generated it is not. Indeed, let C be a countable subset of $c_\lambda^\downarrow \cap A$; by construction we have for every $c \in C$ a $\beta < \lambda$ such that $c \leqslant c_\beta$. Let γ be the supremum of these β's; then $\gamma + 1 < \lambda$ and so $c_{\gamma+1} < c_\lambda$ but no $c \in C$ is above $c_{\gamma+1}$. $\square$

A proof using elementary substructures runs as follows: let $M \prec H(\theta)$ be $\aleph_0$-covering and of cardinality $\aleph_1$, where θ is suitably large, and assume that B and the chain $\{c_\alpha\colon \alpha < \omega_2\}$ belong to M. Next let δ be the ordinal $M \cap \omega_2$; observe that $\operatorname{cf} \delta = \aleph_1$: if $\operatorname{cf} \delta$ were countable then, because M is $\aleph_0$-covering, δ would be the supremum of an element of M and hence in M. Consider the element c_δ. By Lemma 2.4 there is a countable subset T of $c_\delta^\downarrow \cap B \cap M$ that is cofinal in it. There is then (at least) one element a of T such that $\{\alpha < \delta\colon c_\alpha \leqslant a\}$ is cofinal in δ. However, δ is a solution to

$$x \in \omega_2 \quad \text{and} \quad a < c_x$$

hence there must be a solution β in M but then $\beta < \delta$ and $a < c_\beta$, so that $\{a < \delta\colon c_\alpha \leqslant a\}$ is not cofinal in δ.

The reader is invited to supply a proof of the following proposition, which was established in [15] for algebras with the weak Freese–Nation property.

Proposition 2.6. *An $(\aleph_1, \aleph_0)$-ideal algebra contains no $\aleph_2$-Lusin families.*

An $\aleph_2$-Lusin family is a subset A of pairwise disjoint elements with the following property: for every x at least one of the sets $\{a \in A\colon a \leqslant x\}$ or $\{a \in A\colon a \cap x = 0\}$ has size less than $\aleph_2$.

In the Cohen model $\mathcal{P}(\mathbb{N})/fin$ is $(\aleph_1, \aleph_0)$-ideal. We have already indicated that in the Cohen model $\mathcal{P}(\mathbb{N})/fin$ is an $(\aleph_1, \aleph_0)$-ideal algebra. We state it as a separate theorem for future reference.

Theorem 2.7. *Let V be a model of* CH *and let κ be any cardinal. If G is generic on* $\mathrm{Fn}(\kappa, 2)$ *then in $V[G]$ the algebra $\mathcal{P}(\mathbb{N})$ is $(\aleph_1, \aleph_0)$-ideal.*

As it is clear that ω_2 is the union of an increasing sequence of $\aleph_1$-sized subsets the following Proposition, which is Steprāns' Lemma 2.3, now follows.

Proposition 2.8. *In the $\aleph_2$-Cohen model there is an increasing sequence of $\aleph_0$-ideal subalgebras of $\mathcal{P}(\mathbb{N})$, each of size $\aleph_1$, which is continuous at limits of uncountable cofinality and whose union is all of $\mathcal{P}(\mathbb{N})$.*

Generalizing R_ω. The following definition generalizes Parivičenko's property R_ω. After the definition we discuss it more fully and indicate why it is the best possible generalization of R_ω. The new property is actually a cardinal invariant which somehow quantifies some, but not all, of the strength of $\mathrm{MA}_{\mathrm{countable}}$—see Proposition 2.12 and Remark 4.6.

Definition 2.9. For a Boolean algebra B, say that a subset A is $\aleph_0$-ideal complete, if for any two countable subsets S and T of A with $S \perp T$ there is a $b \in B \setminus A$ such that $b^\downarrow \cap A$ is generated by S and $b^\perp \cap A$ is generated by T. We will let $\mathfrak{m}_c(B)$ denote the minimum cardinality of a subset of B that is not $\aleph_0$-ideal complete. Also $\mathfrak{m}_c$ denotes $\mathfrak{m}_c(\mathcal{P}(\mathbb{N})/fin)$.

A remark about the previous definition might be in order. In the definition of $\aleph_0$-ideal completeness the set A is divided into three subsets: A_S, the set of elements a for which there is a finite subset F of S such that $a \leqslant \bigvee F$; the set A_T, defined similarly, and A_r, the rest of A. The element b must effect the same division of A: we demand that $A_S = \{a \in A\colon a < b\}$, $A_T = \{a \in A\colon a < b'\}$ and $A_r = \{a \in A\colon a \nleqslant b$ and $a \nleqslant b'\}$. Observe that one can also write $A_r = \{a \in A\colon b \wedge a \neq 0$ and $b' \wedge a \neq 0\}$; one says that b *reaps* the set A_r. We see that every subset of size less than $\mathfrak{m}_c(B)$ can always be reaped; we shall come back to this in Section 4.

Thus Parovičenko's property R_ω has become the statement that countable subsets are $\aleph_0$-ideal complete, in other words that $\mathfrak{m}_c(B) > \aleph_0$.

Remark 2.10. In Definition 2.9 we explicitly do not exclude the possibility that S or T is finite or even empty. Thus if $\mathfrak{m}_c(B) > \aleph_0$ then there is no countable strictly increasing sequence with 1 as its supremum: for let S be such a sequence and take $T = \{0\}$, then there must apparently be a $b < 1$ such that $a < b$ for all $a \in S$.

Remark 2.11. In the case of $\mathcal{P}(\mathbb{N})/fin$ one cannot relax the requirements on S and T: consider a Hausdorff gap; this is a pair of increasing sequences $\{a_\alpha\colon \alpha \in \omega_1\}$ and $\{b_\alpha\colon \alpha \in \omega_1\}$ such that $a_\alpha \wedge b_\beta = 0$ for all α and β, and for which there is no x such that $a_\alpha \leqslant x$ for all α and $b_\beta \leqslant x^\perp$ for all β. Thus there are cases with $|S| = |T| = \aleph_1$ where no b can be found.

In an $(\aleph_1, \aleph_0)$-ideal algebra with $\mathfrak{m}_c(B) > \aleph_0$ this can be sharpened, as follows. By recursion one can construct a strictly increasing chain $\langle s_\alpha\colon \alpha < \delta\rangle$ in B with $0 < s_\alpha < 1$ for all α, until no further choices can be made. Because B is $(\aleph_1, \aleph_0)$-ideal this must stop before ω_2 and because $\mathfrak{m}_c(B) > \aleph_0$ we have $\operatorname{cf} \delta = \aleph_1$. Thus we have a situation where no b be found with $|S| = \aleph_1$ and $|T| = 1$ (take $T = \{0\}$). This shows that if $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is $(\aleph_1, \aleph_0)$-ideal then the cardinal number $\mathfrak{t}$ (see [6]) is equal to $\aleph_1$.

The following proposition shows why we are interested in $\mathfrak{m}_c$.

Proposition 2.12. $\mathrm{MA}_{\mathrm{countable}}$ *implies that* $\mathfrak{m}_c = \mathfrak{c}$.

Proof. Let A, S and T be given, where, without loss of generality, we assume that S and T are increasing sequences of length ω and $|A| < \mathfrak{c}$. There is a natural countable poset that produces an infinite set b such that $s < b$ and $t < b'$ for all $s \in S$ and $t \in T$: it consists of triples $\langle p, s, t\rangle$, where $p \in \mathrm{Fn}(\omega, 2)$, $s \in S$, $t \in T$ and $s \cap t \subseteq \mathrm{dom}(p)$. The ordering is $\langle p, s, t\rangle \leqslant \langle q, u, v\rangle$ iff $p \supseteq q, s \supseteq u, t \supseteq v$ and if $n \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)$ then $p(n) = 1$ if $n \in u$ and $p(n) = 0$ if $n \in v$.

It is relatively straightforward to determine a family $\mathcal{D}$ of fewer than $\mathfrak{c}$ dense sets so that any $\mathcal{D}$-generic filter produces an element b as required. $\square$

It is well known that $\mathrm{MA}_{\mathrm{countable}}$ holds in any extension by a ccc finite-support iteration whose length is the final value of the continuum and hence in any model obtained by adding $\mathfrak{c}$ or more Cohen reals.

So in the Cohen model $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is an $(\aleph_1, \aleph_0)$-ideal algebra in which $\mathfrak{m}_c$ is $\mathfrak{c}$. Note that this is then consistent with most cardinal arithmetic. However if only $\aleph_2$ Cohen reals are added then this provides our characterizations of $\mathcal{P}(\mathbb{N})/\mathit{fin}$ and $\mathbb{N}^*$ (see also the results 5.3 through 5.5).

Theorem 2.13. *In the $\aleph_2$-Cohen model the algebra $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is characterized by the properties of being an $(\aleph_1, \aleph_0)$-ideal Boolean algebra of cardinality $\mathfrak{c}$ in which $\mathfrak{m}_c$ has value $\mathfrak{c}$.*

The proof is quite straightforward: we use Proposition 2.8 to express any algebra with the properties of the Theorem as the union of a ω_2-chain of $(\aleph_1, \aleph_0)$-ideal subalgebras and we apply $\mathfrak{m}_c = \mathfrak{c}$ to construct an isomorphism between it and $\mathcal{P}(\mathbb{N})/\mathit{fin}$ by recursion. This result and its proof admit a topological reformulation that is quite appealing.

Theorem 2.14. *In the $\aleph_2$-Cohen model $\mathbb{N}^*$ is the unique compact space that is expressible as the limit of an inverse system $\langle\{X_\alpha\colon \alpha < \omega_2\}, \{f_\alpha^\beta\colon \alpha < \beta < \omega_2\}\rangle$ such that*

(1) *each X_α is a compact zero-dimensional space of weight less than $\mathfrak{c}$;*
(2) *for each limit $\lambda < \omega_2$, X_λ is equal to $\varprojlim_{\beta<\lambda} X_\beta$ and $f_\alpha^\lambda = \varprojlim_{\alpha<\beta<\lambda} f_\alpha^\beta$;*

(3) *for each* $\alpha < \beta < \omega_2$, f_α^β *sends zero-set subsets of* X_β *to zero-sets of* X_α (*i.e., clopen sets are sent to* G_δ*-sets*);
(4) *for each* $\alpha < \omega_2$ *and each pair,* C_0, C_1 *of disjoint cozero-sets of* X_α (*possibly empty*), *there are a* $\beta < \omega_2$ *and a clopen subset* b *of* X_β *such that*

$$f_\alpha^\beta(b) = X_\alpha \setminus C_0 \quad \text{and} \quad f_\alpha^\beta(X_\beta \setminus b) = X_\alpha \setminus C_1.$$

Remark 2.15. It is our (subjective) feeling that the $(\aleph_1, \aleph_0)$-ideal property together with $\mathfrak{m}_c$ captures the essence of the behaviour of $\mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathbb{N})/fin$ in the Cohen model. By Theorem 2.13 this is certainly the case for the $\aleph_2$-Cohen model. Evidence in support of our general feeling will be provided in the next section, where we will derive a number of results from "$\mathcal{P}(\mathbb{N})$ is $(\aleph_1, \aleph_0)$-ideal" that were originally derived in the Cohen model. Apparently it is unknown whether these properties characterize $\mathcal{P}(\mathbb{N})/fin$ in Cohen models with $\mathfrak{c} > \aleph_2$.

Other cardinals. We may generalize Definition 2.3 to cardinals other than $\aleph_1$: we can call a Boolean algebra $(\kappa, \aleph_0)$-ideal if the family of κ-sized $\aleph_0$-ideal subalgebras contains a κ-cub, meaning a subfamily closed under unions of chains of length at most κ (but of uncountable cofinality). Similarly we can define B to be $(*, \aleph_0)$-ideal if it is $(\kappa, \aleph_0)$-ideal for every (regular) κ below $|B|$.

The discussion after Lemma 2.2 establishes that every Boolean algebra with the weak Freese–Nation property $(*, \aleph_0)$-ideal and in any Cohen model the algebra $\mathcal{P}(\mathbb{N})/fin$ is $(*, \aleph_0)$-ideal. One can also prove a suitable version of Lemma 2.4.

Lemma 2.16. *Let* B *be an* $(\kappa, \aleph_0)$*-ideal algebra, let* θ *be a suitably large cardinal and let* M *be an elementary substructure of size* κ *of* $H(\theta)$ *that contains* B. *Then* $B \cap M$ *is an* $\aleph_0$*-ideal subalgebra of* B, *provided* M *can be written as* $\bigcup_{\alpha<\kappa} M_\alpha$, *where* $\langle M_\beta \colon \beta \leqslant \alpha \rangle \in M_{\alpha+1}$ *for all* α.

In applications one also needs M to be $\aleph_0$-covering; this is possible only if the structure $([\kappa]^{\aleph_0}, \subseteq)$ has cofinality κ. This accounts for the assumption $\mathrm{cf}[\kappa]^{\aleph_0} = \kappa$ in Theorem 5.6.

3. The axiom "$\mathcal{P}(\mathbb{N})$ is an $(\aleph_1, \aleph_0)$-ideal algebra"

Throughout this section we assume that $\mathcal{P}(\mathbb{N})$ is an $(\aleph_1, \aleph_0)$-ideal algebra and show how useful this can be as an axiom in itself. We fix an $\aleph_1$-cub $\mathcal{A}$ in $[\mathcal{P}(\mathbb{N})]^{\aleph_1}$ that consists of $\aleph_0$-ideal subalgebras.

Mappings onto cubes. In order to avoid additional definitions we state, in the rest of this section, some of the results in their topological, rather than Boolean algebraic, formulations. We shall also use elementary substructures to our advantage; we shall use the phrase 'by elementarity' to indicate that a judicious choice of equation would give the desired result.

The first result we present is due to Baumgartner and Weese [2].

Theorem 3.1. *If X is a compact space with a countable dense set D such that every infinite subset of D contains a converging subsequence, then X does not map onto* $[0,1]^{\omega_2}$.

Proof. If f were a mapping of X onto $[0,1]^{\omega_2}$ then $f[D]$ would be a countable dense subset of $[0,1]^{\omega_2}$ with the same property as D. Therefore we are done once we show that $[0,1]^{\omega_2}$ has no countable dense subset every infinite subset of which contains a converging sequence. So we take a countable dense subset of $[0,1]^{\omega_2}$, which we identify with $\mathbb{N}$, and exhibit an infinite subset of it that does not contain a converging sequence.

To this end we fix a suitably large cardinal θ and consider an $\aleph_1$-sized $\aleph_0$-covering elementary substructure M of $H(\theta)$. We put $\delta = M \cap \omega_2$ and let $c = \mathbb{N} \cap \pi_\delta^{\leftarrow}[[\frac{1}{4}, 1]]$ and $d = \mathbb{N} \cap \pi_\delta^{\leftarrow}[[0, \frac{3}{4}]]$, where, generally, π_α denotes the projection onto the α-th coordinate.

Let $C \in M$ be a countable set such that $c^{\downarrow} \cap M$ is generated by $C_1 = c^{\downarrow} \cap C$; similarly choose a countable element D of M for d and put $D_1 = d^{\downarrow} \cap D$. This can be done because M is $\aleph_0$-covering.

For $x \in C$ let $S_x = \{\alpha\colon x \subseteq^* \pi_\alpha^{\leftarrow}[[\frac{1}{4}, 1]]\}$. Observe that if $\delta \in S_x$ then S_x is cofinal in ω_2 because, apparently, there is then no solution η to $(\eta \in \omega_2) \wedge (\forall \alpha \in S_x)(\alpha < \eta)$ in M and hence not in $H(\theta)$ either. It follows that C_1 is contained in the set

$$C_2 = \{x \in C\colon\ S_x \text{ is cofinal in } \omega_2\},$$

which, by elementarity, is in M. We define T_y, for $y \in D$, in an analogous way and find the set

$$D_2 = \{y \in D\colon\ T_y \text{ is cofinal in } \omega_2\},$$

which is in M and which contains D_1. We claim that the ideal generated by $C_2 \cup D_2$ does not contain a cofinite subset of $\mathbb{N}$.

Indeed, take $x_1, \dots, x_k$ in C_2 and $y_1, \dots, y_k$ in D_2. We can find distinct $\alpha_1, \dots, \alpha_k$, $\beta_1, \dots, \beta_k$ with $\alpha_i \in S_{x_i}$ and $\beta_i \in T_{y_i}$ for all i. The set $U = \bigcap_{i=1}^k (\pi_{\alpha_i}^{\leftarrow}[[0, \frac{1}{4}]) \cap \pi_{\beta_i}^{\leftarrow}[(\frac{3}{4}, 1]])$ is disjoint from $\bigcup_{i=1}^k (x_i \cup y_i)$ and its intersection with $\mathbb{N}$ is infinite.

Because $C_2 \cup D_2$ is countable we can, by elementarity, find an infinite subset a of $\mathbb{N}$ in M that is almost disjoint from every one of its elements. Now if a had an infinite converging subset then, again by elementarity, it would have one, b say, that belongs to M. However, if $b \subseteq^* c$ then $b \subseteq^* x$ for some $x \in C_1$, which is impossible; likewise $b \subseteq^* d$ is impossible. It follows that $\pi_\delta[b]$ does not converge in $[0,1]$. $\square$

Remark 3.2. A careful study of the proof of Theorem 3.1 shows how one can reach δ by a traditional recursion. Build an increasing sequence $\langle A_\alpha\colon \alpha < \omega_1\rangle$ in $\mathcal{A}$ and a sequence $\langle \delta_\alpha\colon \alpha < \omega_1\rangle$ in ω_2 by doing the following at successor steps. Enumerate A_α as $\langle a_\beta\colon \beta < \omega_1\rangle$ and choose, whenever possible, a subset b_β of α_β that converges in $[0,1]^{\aleph_2}$. Next choose, for each $\beta < \omega_1$, a subset d_β as follows: let $C = \{\gamma < \beta\colon\ S_{a_\gamma} \text{ is cofinal in } \omega_2\}$ and $D = \{\gamma < \beta\colon\ T_{a_\gamma} \text{ is cofinal in } \omega_2\}$ (here S_x and T_x are defined as in the proof); as in the proof we can find a nonzero d_β in $(C \cup D)^{\perp}$. Let $A_{\alpha+1}$ be an element of $\mathcal{A}$ that contains

$A_\alpha \cup \{b_\beta\}_{\beta\in\omega_1} \cup \{d_\beta\}_{\beta\in\omega_1}$ and choose $\delta_{\alpha+1}$ so large that $\sup S_a < \delta_{\alpha+1}$ or $\sup T_a < \delta_{\alpha+1}$ whenever $a \in A_{\alpha+1}$ and S_a or T_a is bounded in ω_2. The rest of the proof is essentially the same.

The next result, from [8], provides a nice companion to Theorem 3.1.

We prove the result for the case $\mathfrak{c} = \aleph_2$ only—basically the same proof will work when $\mathfrak{c} = \aleph_n$ for some $n \in \omega$. For larger values of $\mathfrak{c}$ we need assumptions like $\Box$ to push the argument through.

Theorem 3.3 ($2^{\aleph_1} = \mathfrak{c} = \aleph_2$). *If X is compact, separable and of cardinality greater than $\mathfrak{c}$ then X maps onto $I^{\mathfrak{c}}$.*

Proof. Suppose that X is compact and that $\mathbb{N}$ is dense in X. Fix a suitably large cardinal θ and construct an increasing sequence $\langle M_\alpha\colon \alpha < \omega_2\rangle$ of $\aleph_1$-sized elementary substructures of $H(\theta)$ that are $\aleph_0$-covering and where always $\langle M_\beta\colon \beta < \alpha\rangle \in M_{\alpha+1}$; put $M = \bigcup_{\alpha<\omega_2} M_\alpha$. Furthermore by the cardinality assumptions we can ensure that M^ω and M^{ω_1} are subsets of M.

Fix any point x in $X \setminus M$ (because $|M| < |X|$). For each $\alpha < \omega_2$ let $\mathcal{I}_\alpha = \{F \subseteq \mathbb{N}\colon F \in M_\alpha \text{ and } x \notin \operatorname{cl} F\}$. Because $|\mathcal{I}_\alpha| < \mathfrak{c}$ we have $\mathcal{I}_\alpha \in M$ and so by elementarity there is a point $x_\alpha \in X \cap M$ such that $\mathcal{I}_\alpha = \{F \subseteq \mathbb{N}\colon F \in M_\alpha \text{ and } x_\alpha \notin \operatorname{cl} F\}$. Fix a function $f_\alpha : X \to [0,1]$ so that $f_\alpha(x_\alpha) = 0$ and $f_\alpha(x) = 1$, and set $a_\alpha = \{n\colon f_\alpha(n) < \frac{1}{4}\}$ and $b_\alpha = \{n\colon f_\alpha(n) > \frac{3}{4}\}$. There is a $g(\alpha) < \omega_2$ such that $x_\alpha, f_\alpha, a_\alpha$ and b_α belong to $M_{g(\alpha)}$. Finally, fix a cub C in ω_2 such that $\alpha < \lambda$ implies $g(\alpha) < \lambda$ whenever $\lambda \in C$. Set $S = \{\lambda \in C\colon \operatorname{cf}\lambda = \omega_1\}$.

Now apply the Pressing–Down lemma to find a stationary set $T \subseteq S$ and a $\beta \in \omega_2$ so that, for every $\lambda \in T$, each of $a_\lambda^{\downarrow} \cap M_\lambda$, $a_\lambda^{\perp} \cap M_\lambda$, $b_\lambda^{\downarrow} \cap M_\lambda$ and, $b_\lambda^{\perp} \cap M_\lambda$ is generated by a countable subset of M_β.

By induction on $\lambda \in T$ we prove that

$$\bigl\{(a_\alpha, b_\alpha)\colon \alpha \in T \cap \lambda + 1\bigr\}$$

is a dyadic family. In fact, if H and K are disjoint finite subsets of T then

$$\bigcap_{\alpha\in H} a_\alpha \cap \bigcap_{\alpha\in K} b_\alpha$$

is not in the ideal generated by $\mathcal{I}_\beta$. Let $\lambda = \max(H \cup K)$ and suppose first that $\lambda \in K$. Put $y = \bigcap_{\alpha\in H} a_\alpha \cap \bigcap_{\alpha\in K\setminus\{\lambda\}} b_\alpha$; then y is not contained in any member of $\mathcal{I}_\beta$.

Assume there is an $I \in \mathcal{I}_\beta$ such that $y \cap b_\lambda \subseteq I$; then $y \setminus I$ belongs to $M_\lambda \cap b_\lambda^{\perp}$ and hence it is contained in a $c \in M_\beta \cap b_\lambda^{\perp}$. Because $c \perp b_\lambda$ we have $f_\lambda[c] \subseteq [0, \frac{3}{4}]$ and so $x \notin \operatorname{cl} c$ whence $c \in \mathcal{I}_\beta$. We have a contradiction since it now follows that $y \subseteq I \cup c \in \mathcal{I}_\beta$.

Next suppose $\lambda \in H$ and put $y = \bigcap_{\alpha\in H\setminus\{\lambda\}} a_\alpha \cap \bigcap_{\alpha\in K} b_\alpha$; again, y is not contained in any element of $\mathcal{I}_\beta$. Assume there is $I \in \mathcal{I}_\beta$ such that $y \cap a_\lambda \subseteq I$; now $y \setminus I$ belongs to $M_\lambda \cap a_\lambda^{\perp}$ and hence it is contained in a $c \in M_\beta \cap a_\lambda^{\perp}$. Because $c \perp a_\lambda$ we have $f_\lambda[c] \subseteq [\frac{1}{4}, 1]$ and so $x_\lambda \notin \operatorname{cl} c$; because $c \in M_\lambda$ this means $x \in \operatorname{cl} c$ whence $c \in \mathcal{I}_\beta$. Again we have a contradiction because we have $y \subseteq I \cup c \in \mathcal{I}_\beta$.

It now follows that $\Delta\{f_\lambda\colon \lambda \in T\}$ is a continuous map from X into I^T and that the image of X contains $\{0,1\}^T$, which in turn can be mapped onto $[0,1]^T$. $\square$

This result is optimal: in [11] Fedorčuk constructed, in the $\aleph_2$-Cohen model, a separable compact space of cardinality $\mathfrak{c} = 2^{\aleph_1}$ that does not map onto $I^{\mathfrak{c}}$ because its weight is $\aleph_1$.

The size of sequentially compact spaces. The next result arose in the study of compact sequentially compact spaces (see [9] for the applications). Recall that a filter (base) of sets in a space X is said to *converge* to a point if every neighbourhood of the point contains an element of the filter (base).

Lemma 3.4. *If X is a regular space and $\mathbb{N} \subseteq X$ has the property that every infinite subset contains a converging sequence then for each ultrafilter u on $\mathbb{N}$ that converges to some point of X there is an $\aleph_1$-sized filter subbase, v, that converges* (*to the same point*).

Proof. Let u be an ultrafilter on $\mathbb{N}$ that converges to a point x of X. Let $M \prec H(\theta)$ be any $\aleph_1$-sized $\aleph_0$-covering model such that u, X and x are in M. We shall prove that $v = M \cap u$ also converges to x.

Since M is $\aleph_0$-covering and $u \in M$, there is an increasing chain $\{u_\alpha\colon \alpha \in \omega_1\}$ of countable subsets of u such that each u_α is a member of M and $u \cap M = \bigcup\{u_\alpha\colon a \in \omega_1\}$. For each $\alpha \in \omega_1$, there is an $a_\alpha \subseteq \mathbb{N}$ such that $a_\alpha \in M$ and $a_\alpha \setminus U$ is finite for each $U \in u_\alpha$. By the assumption on the embedding of $\mathbb{N}$ in X, we may assume that a_α converges to a point $x_\alpha \in X$. Observe that for each $b \in \mathcal{P}(\mathbb{N}) \cap M$ we have $b \in u$ if and only if $a_\alpha \subseteq^* b$ for uncountably many α.

Suppose that x is an element of some open subset W of X. Let $\{b_n\colon n \in \omega\} \subseteq M \cap \mathcal{P}(\mathbb{N})$ generate $(W \cap \mathbb{N})^\perp \cap M$. Since u converges to x, the set $W \cap \mathbb{N}$ is a member of u. Therefore $\mathbb{N}\setminus b_n$ is a member of u for each n, hence there is an α such that $\{\mathbb{N}\setminus b_n\colon n \in \omega\} \subseteq u_\alpha$. It follows, then, that a_β is almost disjoint from each b_n for all $\beta \geqslant \alpha$. Thus, for $\alpha \leqslant \beta < \omega_1$ we have $a_\beta \notin (W \cap \mathbb{N})^\perp$, which means that $W \cap a_\beta$ is infinite for each $\beta \geqslant \alpha$. It follows that $\{x_\beta\colon \alpha \leqslant \beta < \omega_1\}$ is contained in the closure of W. Since W was an arbitrary neighbourhood of x and X is regular, it follows that there is an α' such that $\{x_\beta\colon \alpha' \leqslant \beta < \omega_1\}$ is contained in W. Since W is open, it follows that a_β is almost contained in W whenever $\alpha' \leqslant \beta < \omega_1$.

Now suppose that $\{c_n\colon n \in \omega\} \subseteq M \cap \mathcal{P}(\mathbb{N})$ generates $(W \cap \mathbb{N})^\downarrow \cap M$. By the above, it follows that, whenever $\alpha' \leqslant \beta < \omega$, there is an n such that a_β is almost contained in c_n. Fix n such that a_β is almost contained in c_n for uncountably many η. As we observed above, it follows that $c_n \in u$. Therefore, as required, we have shown that W contains a member of v. $\square$

Theorem 3.5. *If X is a regular space in which $\mathbb{N}$ is dense and every subset of $\mathbb{N}$ contains a converging sequence, then X has cardinality at most $2^{\aleph_1}$.*

Proof. Each point of X will be the unique limit point of some filter base on $\mathbb{N}$ of cardinality $\aleph_1$. $\square$

Compare this theorem with Theorem 3.1, which draws the conclusion that X cannot be mapped onto $[0,1]^{\aleph_2}$. In fact if $2^{\aleph_1} < 2^{\aleph_2}$ then Theorem 3.1 becomes a consequence of Theorem 3.5.

$\mathbb{N}^*$ *minus a point.* It was shown by Gillman in [16], assuming CH, that for every point u of $\mathbb{N}^*$ one can partition $\mathbb{N}^* \setminus \{u\}$ into two open sets, each of which has u in its closure. Clearly this show that $\mathbb{N}^* \setminus \{u\}$ is not C^*-embedded in $\mathbb{N}^*$. Here we present Malykhin's result, from [20], that establishes the complete opposite.

Theorem 3.6 $(\mathfrak{m}_c > \aleph_1)$. *$\mathbb{N}^*$ minus a point is C^*-embedded in $\mathbb{N}^*$.*

Proof. Assume that $\mathbb{N}^* \setminus \{u\}$ is not C^*-embedded; so there is a continuous function $f : \mathbb{N}^* \setminus \{u\} \to [0,1]$ such that u is simultaneously a limit point of $f^{\leftarrow}(0)$ and $f^{\leftarrow}(1)$.

Fix an increasing sequence $\{c_n : n \in \omega\}$ in $\mathcal{P}(\mathbb{N}) \setminus u$ such that in the case that u is not a P-point every member of u meets some c_n in an infinite set. Now define $\mathcal{I} = \{a \in \{c_n\}_n^{\perp} : a^* \subseteq f^{\leftarrow}(0)\}$ and $\mathcal{J} = \{a \in \{c_n\}_n^{\perp} : a^* \subseteq f^{\leftarrow}(1)\}$. The ideals $\mathcal{I}$ and $\mathcal{J}$ are P-ideals: if I is a countable subset of $\mathcal{I}$ then apply $\mathfrak{m}_c > \aleph_0$ to find $a \in \{c_n\}_n^{\perp}$ with $I \subseteq a^{\downarrow}$. Because $a \notin u$ the function f is defined on all of a^*; it then follows that there is a $b \subseteq a$ such that $I \subseteq b^{\downarrow}$ and $b^* \subseteq f^{\leftarrow}(0)$.

Claim 1. *If $U \in u$ then there is $a \in \mathcal{I}$ such that $a \subseteq U$ (similarly there is $b \in \mathcal{J}$ with $b \subseteq U$).*

Proof of Claim 1. For every n the set $a_n = U \setminus c_n$ belongs to u, hence a_n^* meets $f^{\leftarrow}(0)$ and there is a subset b_n of a_n with $b_n^* \subseteq f^{\leftarrow}(0)$—here we use the well-known fact that $f^{\leftarrow}(0)$ is regularly closed. Now take an infinite set a such that $b \subseteq^* \bigcup_{m \geqslant n} a_m$ for all n; then $a \subseteq U$ and $a^* \subseteq f^{\leftarrow}(0)$.

Let M be an $\aleph_0$-covering elementary substructure of $H(\theta)$, of size $\aleph_1$, that contains u, f and $\{c_n : n \in \omega\}$.

Claim 2. *If $b \in \mathcal{P}(\mathbb{N})/\mathit{fin}$ is such that $\mathcal{I} \cap M \subseteq b^{\downarrow}$ (or $\mathcal{J} \cap M \subseteq b^{\downarrow}$) then there is $U \in u \cap M$ such that $U \subseteq^* b$.*

Proof of Claim 2. Let $C \subseteq b^{\downarrow} \cap M$ be a countable cofinal set and choose for every $c \in C$, whenever possible, an $i_c \in \mathcal{I} \cap M$ such that $c \nleqslant i_c$. Let $I \in M$ be a countable subset of $\mathcal{I}$ that contains all the possible i_c; because I is countable there is $i \in \mathcal{I} \cap M$ such that $a < i$ for all $a \in I$. Note that $i \in b^{\downarrow} \cap M$, hence there is $c \in C$ such that $i < c$; it follows that $\mathcal{I} \cap M \subseteq c^{\downarrow}$. Note that in M there is no solution to "$x \in \mathcal{I}$ and $x \nleqslant c$" hence there is none in $H(\theta)$; it follows that $\mathcal{I} \subseteq c^{\downarrow}$. But this implies that $c \in u$.

The claim implies that if $b \in \mathcal{P}(\mathbb{N})$ meets every $U \in u \cap M$ in an infinite set then there are $I \in \mathcal{I} \cap M$ and $J \in \mathcal{J} \cap M$ that meet b in an infinite set. This in turn implies that the closed set $F = \bigcap\{U^* : U \in u \cap M\}$ is contained in $\operatorname{cl} f^{\leftarrow}(0) \cap \operatorname{cl} f^{\leftarrow}(1)$. The inequality $\mathfrak{m}_c > \aleph_1$

implies that $u \cap M$ does not generate an ultrafilter, so that F consists of more than one point. This contradicts our assumption that u is the only point in $\mathrm{cl}\, f^{\leftarrow}(0) \cap \mathrm{cl}\, f^{\leftarrow}(1)$. $\square$

A first-countable nonremainder. The final result in this section is due to Bell [3]. He produced a compact first countable space which is not a continuous image of $\mathbb{N}^*$ (equivalently: not a remainder of $\mathbb{N}$). We will show that such a space can be taken to be a subspace of the following space, which is an image of $\mathbb{N}^*$. The space is, in hindsight, easy to describe. In the first version of this paper we started out with a generalization of Alexandroff's doubling procedure; the referee rightfully pointed out that we were simply working with the square of the Alexandroff double of the Cantor set. In private correspondence, Bell points out that his original space is not embeddable in the square of the Alexandroff double.

Definition 3.7. Let $\mathbb{D}$ be the Alexandroff double of the Cantor set, i.e., $\mathbb{D} = \mathbb{C} \times 2$, topologized as follows: all points of $\mathbb{C} \times \{1\}$ are isolated and basic neighbourhoods of a point $\langle x, 0\rangle$ is of the form $(U \times 2) \setminus \{\langle x, 1\rangle\}$, where U is a neighbourhood of x in $\mathbb{C}$. It is well known that this results in a compact first countable space.

We let $\mathbb{K} = \mathbb{D} \times \mathbb{D}$. We shall show that $\mathbb{K}$ is a continuous image of $\mathbb{N}^*$ and that it contains a closed subspace that is *not* a continuous image of $\mathbb{N}^*$.

In proving that $\mathbb{K}$ is a continuous image of $\mathbb{N}^*$ we use results from [4]. We let $W = \{\langle k, l\rangle \in \mathbb{N}^2 \colon l \leqslant 2^k\}$ and we let $\pi : W \to \mathbb{N}$ be the projection on the first coordinate. A compact space is called an *orthogonal* image of $\mathbb{N}^*$ if there is a continuous map $f : W^* \to X$ such that the diagonal map $\beta\pi \triangle f : W^* \to \mathbb{N}^* \times X$ is onto. Theorem 2.5 of [4] states that products of $\mathfrak{c}$ (or fewer) orthogonal images of $\mathbb{N}^*$ are again orthogonal images of $\mathbb{N}^*$. Thus the following proposition more than shows that $\mathbb{K}$ is a continuous image of $\mathbb{N}^*$.

Proposition 3.8. *The space $\mathbb{D}$ is an orthogonal image of $\mathbb{N}^*$.*

Proof. Let $\{q_l \colon l \in \mathbb{N}\}$ be a countable dense subset of $\mathbb{C}$ and define $f : W \to \mathbb{C}$ by $f(k,l) = q_l$; observe that βf maps W^* onto $\mathbb{C}$ and that $\beta f(u) = q_l$ for all u in $\{\langle k, l\rangle \colon 2^k \geqslant l\}^*$. This readily implies that $\beta\pi \triangle \beta f$ maps W^* onto $\mathbb{C}$.

A minor modification of the usual argument that nonempty G_δ-subsets of W^* have nonempty interior lets us associate with every $x \in \mathbb{C}$ a subset A_x of W that meets all but finitely many of the vertical lines $V_k = \{\langle k, l\rangle \colon l \leqslant 2^k\}$ and such that $\beta f[A_x^*] = \{x\}$. Now define $g : W \to W$ by $g(k+1, 2l) = g(k+1, 2l+1) = \langle k, l\rangle$ (and $g(0,0) = \langle 0, 0\rangle$); observe that $B_x = g^{\leftarrow}[A_x]$ meets all but finitely many V_k in at least two points, so that we may split it into two parts, C_x and D_x, each of which meets all but finitely many V_k.

Now we turn the map $\beta f \circ \beta g : W^* \to \mathbb{C}$ into a map from W^* to $\mathbb{D}$: every point of D_x^* will be mapped to $\langle x, 1\rangle$ and the points u of $W^* \setminus \bigcup_x D_x^*$ will be mapped to $\langle (\beta f \circ \beta g)(u), 0\rangle$. It is straightforward to verify that the map h thus obtained witnesses that $\mathbb{D}$ is an orthogonal image of $\mathbb{N}^*$. $\square$

Theorem 3.9 ($2^{\aleph_1} = \mathfrak{c}$). *The space $\mathbb{K}$ has a compact subspace X that is not an image of $\mathbb{N}^*$.*

Proof. We obtain X by removing a (suitably chosen) set of isolated points from $\mathbb{K}$. We enumerate $\mathbb{C}$ as $\{r_\alpha\colon \alpha < \mathfrak{c}\}$ and we use our assumption $2^{\aleph_1} = \mathfrak{c}$ to enumerate the family $[\omega_1]^{\aleph_1}$ as $\{A_\alpha\colon \alpha \in \mathfrak{c}\}$ with cofinal repetitions. The set of isolated points that we keep is $\{\langle\langle r_\alpha, 1\rangle, \langle r_\beta, 1\rangle\rangle\colon \alpha \notin A_\beta \text{ or } \beta \notin A_\alpha\}$. Furthermore, we let U_α be intersection of X with the clopen 'cross'

$$\mathbb{D} \times \{\langle r_\alpha, 1\rangle\} \cup \{\langle r_\alpha, 1\rangle\} \times \mathbb{D}.$$

Note that then for all α one has $A_\alpha = \{\xi \in \omega_1\colon U_\xi \cap U_\alpha = \emptyset\}$.

Now suppose that f is a mapping of $\mathbb{N}^*$ onto X and for each $\alpha \in \mathfrak{c}$ fix a representative $a_\alpha \subseteq \mathbb{N}$ for $f^{\leftarrow}[U_\alpha]$. Observe that thus $A_\alpha = \{\xi \in \omega_1\colon a_\xi \cap a_\alpha =^* \emptyset\}$. Fix an $\aleph_1$-sized $\aleph_0$-ideal subalgebra B of $\mathcal{P}(\mathbb{N})/\mathit{fin}$ that contains $\{a_\xi\colon \xi \in \omega_1\}$.

For each $b \in B$ let $S_b = \{\xi\colon a_\xi < b\}$ and pick $\alpha \in \mathfrak{c}$ such that both $S_b \setminus A_\alpha$ and $s_b \cap A_\alpha$ have cardinality $\aleph_1$ whenever S_b has cardinality $\aleph_1$. Now $B \cap a_\alpha^\perp$ is countably generated and it contains the uncountable set $\{a_\xi\colon \xi \in A_\alpha\}$; it follows that there is a $b < a_\alpha^\perp$ such that S_b is uncountable. But now pick any $\xi \in S_b \setminus A_\alpha$. Then $a_\xi \subseteq^* b \subseteq^* a_\alpha^\perp$ yet $a_\xi \cap a_\alpha \neq^* \emptyset$—a clear contradiction. □

4. Other cardinal invariants

In this section we relate $\mathfrak{m}_c(B)$ to other known cardinal invariants of Boolean Algebras; we have already connected $\mathfrak{m}_c$ to the idea of reaping. We formalize this idea in the following definition, which is analogous to the cardinal $\mathfrak{r}$ in $\mathcal{P}(\mathbb{N})/\mathit{fin}$ (see [5,1]).

Definition 4.1. A subset A of a Boolean algebra B is *reaped* by the element $b \in B$, if b and its complement meet every non-zero element of A. The cardinal $\mathfrak{r}(B)$ is defined as the minimum cardinality of a subset A of B that is not reaped by any element of B.

Our discussion after Definition 2.9 therefore establishes the inequality $\mathfrak{r}(B) \geqslant \mathfrak{m}_c(B)$. The other half, so to speak, of $\mathfrak{m}_c$ is provided by the proper analogue, for arbitrary Boolean algebras, of the cardinal number $\mathfrak{d}$.

In [6] van Douwen showed that $\mathfrak{d}$ is equal to the number $\mathfrak{d}_2$ from the following definition.

Definition 4.2. If $D \subseteq \omega^\omega$ and $A \subseteq [\omega]^{\aleph_0}$ then D is said to *dominate on A* if for each $g \in \omega^\omega$ there are $d \in D$ and $a \in A$ such that $g(n) < d(n)$ for each $n \in a$. The cardinal $\mathfrak{d}_2$ is defined as

$$\mathfrak{d}_2 = \min\{|A| + |D|\colon A \subseteq [\omega]^{\aleph_0}, D \subseteq \omega^\omega \text{ and } D \text{ dominates on } A\}.$$

To find a natural analogue of the cardinal invariant $\mathfrak{d}_2$ in a general Boolean algebra we proceed along the lines of Rothberger's work on the cardinals $\mathfrak{b}$ and $\mathfrak{d}$. For this we say that an ideal $\mathcal{I}$ in a Boolean algebra B is *co-generated* by a set S if $\mathcal{I} = S^{\perp}$. We will say that $\mathcal{I}$ is countably co-generated if there is a countably infinite set that co-generates it but, in order to avoid cumbersome consideration of cases, no finite set co-generates it. The cardinal invariant $\mathfrak{d}$ is naturally equal to the minimum cardinal of a cofinal subset of any countably co-generated non-principal ideal in $\mathcal{P}(\mathbb{N})/\mathit{fin}$ and likewise $\mathfrak{b}$ is the cardinal of an unbounded subset of a countably co-generated non-principal ideal in $\mathcal{P}(\mathbb{N})/\mathit{fin}$—this is so because any countably co-generated ideal in $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is naturally isomorphic to the ideal in $\mathcal{P}(\omega \times \omega)/\mathit{fin}$ that is generated by the set of the form

$$L_f = \bigl\{\langle m, n\rangle\colon\ m \in \omega,\ n \leqslant f(m)\bigr\},$$

where $f \in \omega^{\omega}$.

Proposition 4.3. *If $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})/\mathit{fin}$ is a countably co-generated ideal then $\mathfrak{d}_2$ is equal to the minimum cardinality of a subset $\mathcal{J}$ of $\mathcal{I}$ such that no member of $\mathcal{I}$ meets every member of $\mathcal{J}$.*

Proof. Given A and D let $\mathcal{J}$ be the set of graphs $g \restriction a$, where $g \in D$ and $a \in A$. Observe that $g \restriction a$ is disjoint from L_f iff $f(n) < g(n)$ for all $n \in a$.

Conversely, given $\mathcal{J}$ construct for each $J \in \mathcal{J}$ a function g_J with domain $a_J = \{m\colon\ (\exists n)\ (\langle m, n\rangle \in J)\}$ whose graph is contained in J. Clearly if $L_f \cap J = \emptyset$ then $g_J(n) > f(n)$ for $n \in a_J$. $\square$

It is clear that this proof uses a lot of the underlying structure of $\mathcal{P}(\mathbb{N})/\mathit{fin}$; one cannot hope to do something similar for arbitrary Boolean algebras.

Definition 4.4. For a Boolean algebra B, let $\mathfrak{d}_2(B)$ be the minimum cardinal κ such that whenever $\mathcal{I}$ is a countably co-generated ideal of B and A is a subset of $\mathcal{I}$ of cardinality less than κ, there is a $b \in \mathcal{I}$ that meets each member of A.

We get the following characterization of $\mathfrak{m}_c(B)$, with the immediate corollary that $\mathfrak{m}_c = \min\{\mathfrak{r}, \mathfrak{d}\}$.

Theorem 4.5. *If B is a Boolean algebra with $\mathfrak{m}_c(B) > \aleph_0$ then $\mathfrak{m}_c(B)$ is the minimum of $\mathfrak{r}(B)$ and $\mathfrak{d}_2(B)$.*

Proof. We have already seen that $\mathfrak{r}(B) \geqslant \mathfrak{m}_c(B)$.

Let $\mathcal{I}$ be co-generated by the countable set S and let $\mathcal{J}$ be a subset of $\mathcal{I}$ of cardinality less than $\mathfrak{m}_c(B)$. By our assumption on $\mathfrak{m}_c(B)$ the set $A = S \cup J$ has cardinality less than $\mathfrak{m}_c(B)$ as well so that it is $\aleph_0$-ideal complete. There is therefore an element b of B such that $s < b$ for all $s \in S$ and such that b reaps $\mathcal{J}$; then b' is an element of $\mathcal{I}$ that meets all elements of $\mathcal{J}$. Thus $\mathfrak{d}_2(B) \geqslant \mathfrak{m}_c(B)$.

Next let A be a subset of B, of size less than both $\mathfrak{r}(B)$ and $\mathfrak{d}_2(B)$. Let S and T be countable subsets of A with $S \perp T$ and divide A into three subsets: A_S, the set of elements a for which there is a finite subset F of S such that $a \leqslant \bigvee F$; the set A_T, defined similarly, and A_r, the rest of A. Applying $\mathfrak{m}_c(B) > \aleph_0$ we find for each $a \in A_r$ a nonzero element b_a below a that is in $(A_S \cup A_T)^\perp$ and we find $b \in B$ such that $S \subseteq b^\downarrow$ and $T \subseteq b^\perp$. Because $|A| < \mathfrak{d}_2(B)$ we can find $b_1 < b$ and $b_2 < b'$ such that

- $b_1 \perp S$ and for all $a \in A_r$ if $b \cap b_a \neq 0$ then $b_1 \cap b_a \neq 0$, and
- $b_2 \perp T$ and for all $a \in A_r$ if $b' \cap b_a \neq 0$. then $b_2 \cap b_a \neq 0$.

Because $|A| < \mathfrak{r}(B)$ we can find $c_1 < b_1$ that reaps all possible $b_1 \cap b_a$ and likewise we can find $c_2 < b_2$. Finally then $d = b \cap c_1' \cap c_2$ is as required in the definition of $\aleph_0$-ideal completeness. □

Remark 4.6. We can now see that $\mathfrak{m}_c = \mathfrak{c}$ does not imply $\mathrm{MA}_{\text{countable}}$; indeed, it is well known that in the Laver model $\mathrm{MA}_{\text{countable}}$ fails but that also $\mathfrak{d} = \mathfrak{r} = \mathfrak{c}$.

5. General structure of $(\aleph_1, \aleph_0)$-ideal algebras

In this section we explore the general structure of $(\aleph_1, \aleph_0)$-ideal algebras. It is straightforward to check that "being an $\aleph_0$-ideal subalgebra of" is a transitive relation.

Proposition 5.1. *If A is an $\aleph_0$-ideal subalgebra of B and B is an $\aleph_0$-ideal subalgebra of C, then A is an $\aleph_0$-ideal subalgebra of C.*

We have already mentioned Parovičenko's theorem that under CH the algebra $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is the unique $(\aleph_1, \aleph_0)$-ideal algebra with $\mathfrak{m}_c = \mathfrak{c}$. This leads us to the following definition.

Definition 5.2. A Boolean algebra B is Cohen–Parovičenko if B is $(*, \aleph_0)$-ideal and $\mathfrak{m}_c(B) = \mathfrak{c}$.

In the special case when $\mathfrak{c} = \aleph_2$ we have a convenient characterization in terms of well-orderings at our disposal.

Proposition 5.3. *If $\mathfrak{c} = \aleph_2$ then an algebra B of cardinality $\mathfrak{c}$ is Cohen–Parovičenko if and only if for each enumeration $B = \{b_\alpha \colon \alpha \in \omega_2\}$ the set of $\lambda \in \omega_2$ for which $B_\lambda = \{b_\alpha \colon \alpha \in \lambda\}$ is both an $\aleph_0$-ideal and an $\aleph_0$-ideal complete subalgebra is closed and unbounded in ω_2.*

The reason for adding the prefix 'Cohen' is contained in the following proposition, which together with Theorem 5.5 gives a 'factorization' of Steprāns' characterization of $\mathcal{P}(\mathbb{N})/\mathit{fin}$ (Theorem 2.13). The proposition itself combines Theorem 2.7 and Proposition 2.12.

Proposition 5.4. *In the Cohen model $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is Cohen–Parovičenko.*

The following theorem combines Parovičenko's and Steprāns' theorems into one. We have been unable to find any sort of similar result in the case that $\mathfrak{c} > \aleph_2$.

Theorem 5.5. *If $\mathfrak{c} \leqslant \aleph_2$, then all Cohen–Parovičenko Boolean algebras of cardinality $\mathfrak{c}$ are pairwise isomorphic.*

To show that this theorem is never vacuous we now construct $(*, \aleph_0)$-ideal algebras with prescribed $\mathfrak{m}_c$-numbers, including $\mathfrak{c}$. Thus we see that the idealness of an algebra has no bearing on its $\mathfrak{m}_c$-number. By contrast, a careful inspection of [13, Proposition 2.3 and Corollary 2.4] will reveal that if $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is $(\kappa, \aleph_0)$-ideal and $\mathrm{cf}[\kappa]^{\aleph_0} = \kappa$ then $\mathfrak{m}_c > \kappa$, thus showing that $\mathfrak{m}_c = \mathfrak{c}$ in case $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is $(*, \aleph_0)$-ideal and there are cofinally many κ below $\mathfrak{c}$ with $\mathrm{cf}[\kappa]^{\aleph_0} = \kappa$.

Theorem 5.6. *For each regular cardinal $\kappa < \mathfrak{c}$ such that $\mathrm{cf}[\kappa]^{\aleph_0} = \kappa$, there is an algebra of cardinality $\mathfrak{c}$ that is $(*, \aleph_0)$-ideal and has κ as its $\mathfrak{m}_c$-number. Thus, if $\mathfrak{c}$ is regular then there is Cohen–Parovičenko algebra of cardinality $\mathfrak{c}$.*

We split the construction into two propositions.

Proposition 5.7. *There is a $(*, \aleph_0)$-ideal algebra B of size $\mathfrak{c}$ with $\mathfrak{m}_c(B) \geqslant \kappa$.*

Proof. We obtain B as the direct limit of a sequence $\langle B_\xi \colon \xi < \mu \rangle$, where μ is the ordinal $\mathfrak{c} \cdot \kappa$ if $\kappa < \mathfrak{c}$ and $\mu = \mathfrak{c}$ otherwise.

We begin by letting B_0 be the two-element algebra. At limit stages ξ set

$$B_\xi = \varinjlim_{\eta < \xi} B_\eta.$$

Carry an enumeration, $\{\langle S_\xi, T_\xi \rangle \colon \xi \in \mu\}$ with cofinal repetitions, of pairs of countably infinite subsets of B so that $S_\xi \cup T_\xi \subseteq B_\xi$ and $S_\xi \perp T_\xi$ for all ξ. For simplicity, assume that S_ξ and T_ξ are strictly increasing sequences (if infinite) or singletons (if finite, where an empty set may always be replaced by $\{0\}$).

To construct $B_{\xi+1}$ from B_ξ first take the completion $\widetilde{B}_\xi$ of B_ξ and in it we define s_ξ and t_ξ by $s_\xi = \bigvee S_\xi$ and $t'_\xi = \bigvee T_\xi$; note that $s_\xi \leqslant t_\xi$. There are two cases to consider.

If $s_\xi < t_\xi$ then we let $B_{\xi+1}$ be the subalgebra of $\widetilde{B}_\xi^2$ generated by the diagonal $\{\langle b, b \rangle \colon b \in B_\xi\}$ and the element $b_\xi = \langle s_\xi, t_\xi \rangle$. Observe that $\langle s_\xi, t_\xi \rangle$ does exactly what is required in Definition 2.9—with $A = B_\xi$, $S = S_\xi$ and $T = T_\xi$. Also observe that B_ξ is an $\aleph_0$-ideal subalgebra of $B_{\xi+1}$: a typical element b of $B_{\xi+1}$ looks like $(b_\xi \wedge a_0) \vee (b'_\xi \wedge a_1)$ and from this it is easily seen that the countable set $A_b = \{(s \wedge a_0) \vee (t \wedge a_1) \vee (a_0 \wedge a_1) \colon s \in S_\xi,\ t \in T_\xi\}$ generates $b^\downarrow \cap B_\xi$: if $a \leqslant b$ then $a \leqslant a_0 \vee a_1$ and so we have to cover $a \wedge a'_1$ (which is below $b_\xi \wedge a_0$), $a \wedge a'_0$ (which is below $b'_\xi \wedge a_1$) and $a \wedge a_0 \wedge a_1$.

If $s_\xi = t_\xi$ then this still works if S_ξ and T_ξ are both infinite or both finite but not if, say, S_ξ is infinite and T_ξ is finite, for then we seek an element b_ξ such that $s < b_\xi < t_\xi$ for all $s \in S_\xi$—note that in this case $T_\xi = \{t'_\xi\}$ and that t_ξ belongs to B_ξ. To remedy this we take the Stone space X_ξ of B_ξ and consider the closed set $C_\xi = s_\xi \setminus \bigcup S_\xi$. We let $B_{\xi+1}$ be the clopen algebra of the subspace $Y_\xi = (X_\xi \times \{0\}) \cup (C_\xi \times \{1\})$ of $X_\xi \times \{0, 1\}$. Observe that $b_\xi = s_\xi \times \{0\}$ does what we want: for every $s \in S_\xi$ we have $s < b_\xi < t_\xi$, because t_ξ now corresponds to $b_\xi \cup (C_\xi \times \{1\})$. A typical element b of

$B_{\xi+1}$ now looks like $(b_\xi \wedge a_0) \vee (c_\xi \wedge a_1) \vee (t'_\xi \wedge a_2)$, where $c_\xi = C_\xi \times \{1\}$—because $C_\xi \subseteq s_\xi$ we have $b \in B_\xi$ iff we can take $a_0 = a_1$. As above, we can verify that $A_b = \{(s \wedge a_0) \vee (t_\xi \wedge a_1) \vee (t'_\xi \wedge a_2)\colon s \in S_\xi\}$ generates $b^\downarrow \cap B_\xi$. If $a \leqslant b$ then $t'_\xi \wedge a \leqslant t_\xi \wedge a_2$, so we concentrate on $a^\dagger = a \wedge t_\xi = (a \wedge b_\xi) \vee (a \wedge c_\xi)$. Now $a^\dagger \wedge a'_1 \leqslant b_\xi$, so there is an $s \in S_\xi$ above $a^\dagger \wedge a'_1$. For this s we have $a \leqslant (s \wedge a_0) \vee (t_\xi \wedge a_1) \vee (t'_\xi \wedge a_2)$.

We shall refer to ξ as of type 0 if we simply adjoin $\langle s_\xi, t_\xi\rangle$; the other ξ will be of type 1.

It is straightforward to check that $\mathfrak{m}_c(B) \geqslant \kappa$: if $|A| < \kappa$ then $A \subseteq B_\eta$ for some η and if S and T are countable subsets of A with $S \perp T$ then there is a ξ above η with $\langle S, T\rangle = \langle S_\xi, T_\xi\rangle$; the element b_ξ is as required for A, S and T.

We show that B is $(*, \aleph_0)$-ideal by showing that $M \cap B$ is $\aleph_0$-ideal in B whenever M is an elementary substructure of $H(\mathfrak{c}^+)$ with $\langle\langle S_\xi, T_\xi\rangle\colon \xi < \mu\rangle$ and $\langle B_\xi\colon \xi < \mu\rangle$ both in M, and with $|M|$ less than $\mathfrak{c}$ and regular.

Let $b \in B \setminus M$, take $e \in b^\downarrow \cap M$, and fix the δ and ξ for which $b \in B_{\delta+1} \setminus B_\delta$ and $e \in B_{\xi+1} \setminus B_\xi$ respectively.

Claim 1. *If $\xi \neq \delta$ then there is an $a \in A_b$ with $e \leqslant a$.*

Proof of Claim 1. If $\xi < \delta$ then $e \in B_\delta$ and we are done.

If $\xi > \delta$ we consider two cases. If ξ is of type 0 and $e = (b_\xi \wedge e_0) \vee (b'_\xi \wedge e_1)$ then $e_0 \wedge b' \in b_\xi^\perp$, hence $e_0 \wedge b' \leqslant t$ for some t in T_ξ; likewise $e_1 \wedge b' \leqslant s$ for some s in S_ξ. But then $e \leqslant (e_0 \wedge t') \vee (e_1 \wedge s') \leqslant b$, where the middle element belongs to B_ξ; it follows that there is an $a \in A_b$ with $e \leqslant a$.

If ξ is of type 1 and $e = (b_\xi \wedge e_0) \vee (c_\xi \wedge e_1) \vee (t'_\xi \wedge e_2)$ then $t'_\xi \wedge e_2$ belongs to B_ξ, so we concentrate on the other parts of e—and we assume $e_0, e_1 \leqslant t_\xi$. Observe that $b_\xi \wedge e_0 \leqslant t_\xi \wedge e_0 \leqslant b$: use the fact that $b \in B_\xi$. Next $e_1 \wedge b' \leqslant b_\xi$, so that there is $s \in S_\xi$ with $e_1 \wedge b' \leqslant s$ and hence $c_\xi \wedge e_1 \leqslant s' \wedge e_1 \leqslant b$. We see that $e \leqslant (t_\xi \wedge e_0) \vee (s' \wedge e_1) \vee (t'_\xi \wedge e_2) \leqslant b$, where the middle element belongs to B_ξ; again we can find our $a \in A_b$ with $e \leqslant a$.

This claim essentially takes care of the case $\delta \notin M$: by our obvious inductive assumption we have for every $a \in A_b$ a countable generating set C_a for $a^\downarrow \cap M$. By the claim the countable set $C_b = \bigcup_{a \in A_b} C_a$ generates $b^\downarrow \cap M$ (note that $\xi \in M$, so $\xi \neq \delta$).

To fully finish the proof we must show what to do if $\delta \in M$. The set C_b still takes care of the e with $\xi \neq \delta$. The following two claims show what to add to C_b in order to take care of the e with $\xi = \delta$.

Claim 2. *If δ is of type 0, $e = (b_\delta \wedge e_0) \vee (b'_\delta \wedge e_1)$ and $b = (b_\delta \wedge a_0) \vee (b'_\delta \wedge a_1)$ then there are $c_0 \in C_{a_0}$ and $c_1 \in C_{a_1}$ such that $e \leqslant (b_\delta \wedge c_0) \vee (b'_\delta \wedge c_1) \leqslant b$.*

Proof of Claim 2. Simply observe that $b_\delta \wedge e_i \in a_i^\downarrow \cap M$ for $i = 0, 1$.

We see that we must add $\{(b_\delta \wedge c_0) \vee (b'_\delta \wedge c_1)\colon c_0 \in C_{a_0}, c_1 \in C_{a_1}\}$ to C_b.

Claim 3. *If δ is of type 1 and $e = (b_\delta \wedge e_0) \vee (c_\delta \wedge e_1) \vee (t'_\delta \wedge e_2)$ and $b = (b_\delta \wedge a_0) \vee (c_\delta \wedge a_1) \vee (t'_\delta \wedge a_2)$. Then there are $c_0 \in C_{a_0},\ c_1 \in C_{a_1}$ and $c_2 \in C_{a_2}$ such that $e \leqslant (b_\delta \wedge c_0) \vee (c_\delta \wedge c_1) \vee (t'_\delta \wedge c_2) \leqslant b$.*

Proof of Claim 3. Simply observe that $b_\delta \wedge e_0 \in a_0^{\downarrow} \cap M$, $c_\delta \wedge e_1 \in a_1^{\downarrow} \cap M$ and $t'_\delta \wedge e_2 \in a_2^{\downarrow} \cap M$. Now add $\{(b_\delta \wedge c_0) \vee (c_\delta \wedge c_1) \vee (t'_\delta \wedge c_2)\colon\ c_0 \in C_{a_0},\ c_1 \in C_{a_1},\ c_2 \in C_{a_2}\}$ to C_b. $\square$

Note that if $\kappa = \mathfrak{c}$ we are done: the algebra B is $(*, \aleph_0)$-ideal with $\mathfrak{m}_c(B) = \mathfrak{c}$. In the case where $\kappa < \mathfrak{c}$ we use the cofinality assumption to find a subalgebra of B with the right properties.

Proposition 5.8. *If $\kappa < \mathfrak{c}$ then the algebra B constructed in the proof of Proposition* 5.7 *contains an algebra A of cardinality $\mathfrak{c}$ with $\mathfrak{m}_c(A) = \kappa$.*

Proof. We fix a cofinal subfamily $\{Y_\alpha\colon\ a < \kappa\}$ of $[\kappa]^{\aleph_0}$ with $Y_\alpha \subseteq \alpha$ for all α. We also assume that, for every $\alpha < \kappa$, all ordered pairs $\langle S, T\rangle$ with $S, T \subseteq B_{\mathfrak{c}\cdot\alpha}$ occur in the list $\{\langle S_\xi, T_\xi\rangle\colon\ \mathfrak{c}\cdot\alpha \leqslant \xi < \mathfrak{c}\cdot(\alpha+1)\}$. This enables us to choose, recursively, $\lambda_\alpha \in [\mathfrak{c}\cdot\alpha, \mathfrak{c}\cdot(\alpha+1))$ such that $S_{\lambda_a} = \{b_{\lambda_\beta}\colon\ \beta \in Y_\alpha\}$ and $T_{\lambda_a} = \{0\}$. Note that then $b_{\lambda_\beta} < b_{\lambda_\alpha}$ whenever $\beta \in Y_\alpha$. In what follows we abbreviate b_{λ_α} by p_α.

Because the Y_α form a cofinal family in $[\kappa]^{\aleph_0}$, the family $\{p_\alpha\colon\ \alpha < \kappa\}$ is $\aleph_0$-directed, i.e., if $F \subseteq \kappa$ is countable then there is an α such that $p_\beta < p_\alpha$ for all $\beta \in F$. It follows that $I = \{b\colon\ (\exists a)\ (b \leqslant p_\alpha)\}$ is a P-ideal. We set $F = \{b'\colon\ b \in I\}$ and consider the subalgebra $A = I \cup F$ of B. It is clear that $\mathfrak{m}_c(A) \leqslant \kappa$: no element of A reaps the family $\{p_\alpha\colon\ a < \kappa\}$.

To show $\mathfrak{m}_c(A) \geqslant \kappa$ we take a subalgebra D of A of size less than κ and countable subsets S and T of D with $S \perp T$; we assume S and T are increasing sequences. Also, fix $\alpha < \kappa$ such that $D \subseteq B_{\mathfrak{c}\cdot\alpha}$ and for every $d \in D$ there is $\beta < \alpha$ with $d \leqslant p_\beta$ or $d' \leqslant p_\beta$. If some member of S or T belongs to F then any $b \in B$ that witnesses this instance of $\aleph_0$-ideal completeness of D in B automatically belongs to A.

In the other case, when $S \cup T \subseteq I$, we can assume that Y_α contains, for every $a \in S \cup T$, a β such that $a \leqslant p_\beta$; but then $S \cup T \subseteq p_\alpha^{\downarrow}$. Also note that p_α meets every nonzero element of B_{λ_α} and hence of D. Now choose $\zeta \in [\mathfrak{c}\cdot\alpha, \mathfrak{c}\cdot(\alpha+1))$ such that $S_\zeta = S \cup \{p'_\alpha\}$ and $T_\zeta = T$. Let $d \in D \cap b_\zeta^{\downarrow}$; there is an $s \in S$ with $d \leqslant s \vee p'_\alpha$, then $d \wedge s' \leqslant p'_\alpha$ and so $d \wedge s' = 0$ whence $d \leqslant s$. We see that S generates $b_\zeta^{\downarrow} \cap D$ and, similarly, that T generates $b_\zeta^{\perp} \cap D$.

We finish by showing that A is $(*, \aleph_0)$-ideal. The notation $b^{\downarrow}$ will always mean the set computed in B. Let M be any elementary substructure of $H(\mathfrak{c}^+)$ of regular size less than $\mathfrak{c}$ such that $\langle B_\xi\colon\ \xi < \mu\rangle$ and $\langle \lambda_\alpha\colon\ \alpha \in \kappa\rangle$ are members of M; this ensures that $M \cap B$ is an $\aleph_0$-ideal subalgebra of B. We shall show that for any b in B, the ideal $b^{\downarrow} \cap (M \cap A)$ is countably generated; we denote the countable generating set, when found, by $b^{M,A}$.

Fix $\delta < \mu$ so that $b \in B_{\delta+1} \setminus B_\delta$ and assume we have found $a^{M,A}$ for all $a \in B_\delta$. By Claim 1 in the proof of Proposition 5.7 the set $\bigcup_{a \in A_b} a^{M,A}$ takes care of all $e \in b^{\downarrow} \cap (M \cap$

A), except possibly those in $B_{\delta+1}\backslash B_\delta$. In particular we can set $b^{M,A} = \bigcup_{a\in A_b} a^{M,A}$ when $\delta \notin M$.

Thus we are left with the case where $\delta \in M$. If there is an $e \in b^\downarrow \cap M \cap F$ then $C_b \cap F$ generates $b^\downarrow \cap M \cap A$, where C_b is as defined in the proof of Proposition 5.7. In the other case, where $b^\downarrow \cap M \cap F = \emptyset$, we add

$$\bigl\{(b_\delta \wedge c_0) \vee \bigl(b'_\delta \wedge c_1\bigr)\colon\ c_0 \in a_0^{M,A},\ c_1 \in a_1^{M,A}\bigr\}$$

to $b^{M,A}$ if δ is of type 0 and we add

$$\bigl\{(b_\delta \wedge c_0) \vee (c_\delta \wedge c_1) \vee \bigl(t'_\delta \wedge c_2\bigr)\colon\ c_0 \in a_0^{M,A},\ c_1 \in a_1^{M,A},\ c_2 \in a_2^{M,A}\bigr\}$$

if δ is of type 1.

Indeed, if $e \in b^\downarrow \cap (M \cap A)$ then e belongs to $I \cap M$ and hence so do $e \wedge b_\delta$ and $e \wedge b'_\delta$. Note that $e \wedge b_\delta \leqslant a_0$ and $e \wedge b'_\delta \leqslant a_1$ so that there are $c_0 \in a_0^{M,A}$ and $c_1 \in a_1^{M,A}$ with $e \wedge b_\delta \leqslant c_0 \leqslant a_0$ and $e \wedge b'_\delta \leqslant c_1 \leqslant a_1$ respectively.

If δ is of type 1 then we observe that $e \wedge b_\delta, e \wedge c_\delta$ and $e \wedge t'_\delta$ all belong to $I \cap M$ and are below a_0, a_1 and a_2 respectively. □

Mapping F-spaces onto $\beta\mathbb{N}$. Every compact F-space contains a copy of $\beta\mathbb{N}$: it follows straight from the definition of F-space that the closure of a countably infinite relatively discrete subset is homeomorphic to $\beta\mathbb{N}$. Thus, in a manner of speaking, $\beta\mathbb{N}$ is a minimal F-space. Bell has asked whether $\beta\mathbb{N}$ is also minimal in the mapping-onto sense: does every infinite compact zero-dimensional F-space map onto $\beta\mathbb{N}$? The ease with which $\beta\mathbb{N}$ can be embedded into such a space belies the dual difficulty in constructing an embedding of $\mathcal{P}(\mathbb{N})$ into its algebra of clopen sets. Indeed, we show by means of a Cohen–Parovičenko algebra that such an embedding does not always exist. Before that we prove that Bell's question has a positive answer if the Continuum Hypothesis is assumed.

Proposition 5.9 (CH). *Every infinite compact zero-dimensional F-space maps onto $\beta\mathbb{N}$.*

Proof. It suffices to prove that $\mathcal{P}(\mathbb{N})$ will embed into B where B is infinite and has no (ω,ω)-gaps. Fix any sequence $\{b_n\colon n \in \omega\}$ of pairwise disjoint non-zero elements of B. Let $\{a_\alpha\colon \alpha \in \omega_1\}$ be an enumeration of $\mathcal{P}(\mathbb{N})$ so that $a_n = \{n\}$ for each $n \in \omega$. Inductively choose elements $b_\alpha \in B$ so that the mapping $a_\alpha \to b_\alpha$ lifts to an isomorphism from the algebra generated by $\{a_\beta\colon \beta \leqslant \alpha\}$. If a_α is in the algebra generated by its predecessors then there is nothing to do. Otherwise, by the inductive hypothesis, the ideal $\mathcal{I}$ generated by $\{b_\beta\colon a_\beta < a_\alpha\}$ is disjoint from the ideal $\mathcal{J}$ generated by $\{b_\beta\colon a_\beta \wedge a_\alpha = 0\}$. Since B has no (ω,ω)-gaps, there is a $b_\alpha \in B$ such that $\mathcal{I} \subseteq b_\alpha^\downarrow$ and $\mathcal{J} \subseteq b_\alpha^\perp$. □

Theorem 5.10. *It is consistent that there is an infinite compact zero-dimensional F-space that does not map onto $\beta\mathbb{N}$.*

Proof. It is consistent with $\mathfrak{c} = \aleph_2$ that $\mathcal{P}(\mathbb{N})/\mathit{fin}$ contains an ω_2-chain, this happens, e.g., if MA holds. But now let B be the Cohen–Parovičenko algebra from Theorem 5.6. Clearly

$S(B)$ is a compact zero-dimensional F-space. Assume that f maps $S(B)$ onto $\beta\mathbb{N}$ and let $b_n = f^{\leftarrow}(n)$ for each n. Now let $\mathcal{I}$ be the ideal in B generated by $\{b_n\colon n \in \omega\}$. By the forthcoming Corollary 5.12 $B/\mathcal{I}$ is still $(\aleph_1, \aleph_0)$-ideal and so, by Proposition 2.5, does not contain an ω_2-chain. However $B/\mathcal{I}$ is isomorphic to the algebra of clopen subsets of the closed set $X \setminus \bigcup_n b_n = f^{\leftarrow}(\mathbb{N}^*)$ and certainly does contain ω_2-chains. $\square$

This proof does not work in the $\aleph_2$-Cohen model, where $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is *the* Cohen–Parovičenko algebra. We therefore ask, also in the hope of establishing the consistency with $\neg$CH of a yes answer to Bell's question, the following.

Question 1. Is it true in the $\aleph_2$-Cohen model that every compact zero-dimensional F-space does map onto $\beta\mathbb{N}$?

Quotient algebras. Under CH one can use Parovičenko's theorem to find many copies of $\mathbb{N}^*$ inside of $\mathbb{N}^*$: the proof usually boils down to showing that a quotient of $\mathcal{P}(\mathbb{N})/\mathit{fin}$ by some ideal is isomorphic to $\mathcal{P}(\mathbb{N})/\mathit{fin}$. The same can be done in the Cohen model because many quotients of Cohen–Parovičenko algebras are again Cohen–Parovičenko. First we consider quotients by small ideals.

Lemma 5.11. *If B is a Boolean algebra, A is an $\aleph_0$-ideal subalgebra, and $\mathcal{I}$ is an ideal which is generated by $\mathcal{I} \cap A$, then $A/\mathcal{I}$ is an $\aleph_0$-ideal subalgebra of $B/\mathcal{I}$.*

Proof. Fix any $b \in B$ and fix a cofinal sequence $\{a_n\colon n \in \omega\} \subseteq b^{\downarrow} \cap A$. Let $c \in A$ be such that $c/\mathcal{I} < b/\mathcal{I}$, which means that $c \setminus b$ is covered by some member d of $\mathcal{I} \cap A$. It follows then that $c \setminus d < b$. Hence there is n such that $c \setminus d < a_n < b$. But now it follows that $c/\mathcal{I} < a_n/\mathcal{I}$. $\square$

Corollary 5.12. *If $\mathcal{I}$ is an $\aleph_1$-generated ideal in a $(\kappa, \aleph_0)$-ideal Boolean algebra B, then $B/\mathcal{I}$ is also a $(\kappa, \aleph_0)$-ideal Boolean algebra.*

Another interesting consequence is that ω_1^* is not the image of the Stone space of an $(\aleph_1, \aleph_0)$-ideal algebra.

Corollary 5.13. *The algebra $P(\omega_1)/\mathit{fin}$ cannot be embedded into an $(\aleph_1, \aleph_0)$-ideal algebra.*

Proof. This proceeds much as the proof of Theorem 5.10 since $P(\omega_1)/\mathit{ctble}$ is a quotient of $P(\omega_1)/\mathit{fin}$ by an $\aleph_1$-generated ideal and contains ω_2-chains. $\square$

Corollary 5.14 ($\neg$CH). *If B is Cohen–Parovičenko and $\mathcal{I}$ is an $\aleph_1$-generated ideal then $B/\mathcal{I}$ is again Cohen–Parovičenko.*

Proof. It remains only to prove that $\mathfrak{m}_c(B/\mathcal{I}) = \mathfrak{c}$. To do so, fix countable subsets S and T of B so that $s \wedge t \in \mathcal{I}$ for each $s \in S$ and $t \in T$. Since S and T are countable it is routine to recursively remove from each member of S and T some member of $\mathcal{I}$ so as to ensure that $s \wedge t = 0$ for each $s \in S$ and $t \in T$. Now suppose that A is a subalgebra of B that contains S and T and has cardinality less than $\mathfrak{c}$. We may assume that A contains a generating set for $\mathcal{I}$. Since B is Cohen–Parovičenko there is a $b \in B$ such that S generates $b^{\downarrow} \cap A$ and T generates $b^{\perp} \cap A$. Now suppose $a/\mathcal{I}$ is below $b/\mathcal{I}$, i.e., $a \setminus b \in \mathcal{I}$. Since $A \cap \mathcal{I}$ generates $\mathcal{I}$, there is a $c \in A \cap \mathcal{I}$ such that $a \setminus b < c$. Thus $a \setminus c < b$ and so there is a finite join, s, of members of S such that $a \setminus c < s$. It follows that $\{s/\mathcal{I}\colon s \in S\}$ generates $(b/\mathcal{I})^{\downarrow} \cap A/\mathcal{I}$. Similarly $(b/\mathcal{I})^{\perp} \cap A/\mathcal{I}$ is generated by $\{t/\mathcal{I}\colon t \in T\}$. $\square$

Another situation that occurs frequently is that one has a lifting for the ideal $\mathcal{I}$, this is a Boolean homomorphism $l : B/\mathcal{I} \to B$ with the property that $l(b/\mathcal{I})/\mathcal{I} = b/\mathcal{I}$. In dual terms this means that the closed set $F = S(B) \setminus \bigcup\{i^*\colon i \in \mathcal{I}\}$ is a retract of $S(B)$. The retraction r and the lifting l are connected by the formula $l(b/\mathcal{I}) = r^{\leftarrow}[b^* \cap F]$ for each $b \in B$.

Theorem 5.15. *If $\mathcal{I}$ is an ideal on B for which there is a lifting $l : B/\mathcal{I} \to B$ then for each $\aleph_0$-ideal subalgebra A of B such that $l[A/\mathcal{I}| \subseteq A$ the quotient $A/\mathcal{I}$ is an $\aleph_0$-ideal subalgebra of $B/\mathcal{I}$. Therefore, if B is an $(\kappa, \aleph_0)$-ideal Boolean algebra, then so is $B/\mathcal{I}$.*

Proof. Let A be an $\aleph_0$-ideal subalgebra of B such that $l[A/\mathcal{I}] \subseteq A$. Fix any $b \in B$. We will show that $(b/\mathcal{I})^{\downarrow} \cap A/\mathcal{I}$ is countably generated. In fact, suppose that $\{a_n\colon n \in \omega\}$ generates $l(b/\mathcal{I})^{\downarrow} \cap A$. Fix any $x \in A$ such that $x/\mathcal{I} < b/\mathcal{I}$. By assumption, $x^{\dagger} = l(x/\mathcal{I})$ is also a member of A. Furthermore $l(x/\mathcal{I}) \leqslant l(b/\mathcal{I})$, hence there is an n such that $x^{\dagger} \leqslant a_n$. Clearly then $x/\mathcal{I} = x^{\dagger}/\mathcal{I} \leqslant a_n/\mathcal{I}$. $\square$

6. Other remainders and applications to $\mathbb{N}^*$

We say that a zero-dimensional space K is Cohen–Parovičenko if its algebra of clopen sets is Cohen–Parovičenko. In this section we are interested in identifying which remainders of σ-compact locally compact spaces can be Cohen–Parovičenko; by 'remainder' we mean the Čech–Stone remainder $\beta X \setminus X$—commonly denoted by X^*. We then apply this information and the results of the previous section to the study of $\mathbb{N}^*$ under the assumption that it is Cohen–Parovičenko. We are motivated by the somewhat classical results about $\mathbb{N}^*$ that are known to follow from CH (see [21]). The predisposition of this section is to assume that $\mathbb{N}^*$ is Cohen–Parovičenko and to determine how this affects the structure of $\mathbb{N}^*$ and of other remainders.

In what follows, whenever X is a zero-dimensional compact space, we write $\mathfrak{m}_c(X)$, $\mathfrak{r}(X)$ and $\mathfrak{d}_2(X)$ for the values that these functions have on the Boolean algebra $\mathrm{CO}(X)$ of clopen subsets of X. We first prove a lemma concerning the behaviour of $\mathfrak{d}_2$ and $\mathfrak{r}$ under continuous mappings.

Lemma 6.1. *If $f : X \to Y$ is an open continuous surjection then $\mathfrak{d}_2(Y) \geqslant \mathfrak{d}_2(X)$ and $\mathfrak{r}(Y) \leqslant \mathfrak{r}(X)$.*

Proof. Let $\mathcal{I}$ be an ideal of $\mathrm{CO}(Y)$, co-generated by the family $\{c_n\colon n \in \omega\}$, and let A be a subfamily of $\mathcal{I}$ of size less than $\mathfrak{d}_2(X)$. In $\mathrm{CO}(X)$ we can find an element b such that $b \cap f^{\leftarrow}[c_n] = \emptyset$ for all n and $b \cap f^{\leftarrow}[a] \neq \emptyset$ for all $a \in A$. Because the map f is open the set $f[b]$ is clopen, it also belongs to $\mathcal{I}$ and it meets every element of A.

Next let $\mathcal{C}$ be a family of clopen subsets of X, of size less than $\mathfrak{r}(Y)$. Because f is open the family $\{f[c]\colon c \in \mathcal{C}\}$ consists of clopen sets and so we can find $b \in \mathrm{CO}(Y)$ that reaps it. Then $f^{\leftarrow}[b]$ reaps the family $\mathcal{C}$. □

Our first result is somewhat surprising. It implies that if CH fails then most remainders are not Cohen–Parovičenko. Recall that a space is *basically disconnected* if each cozero-set has clopen closure—dually: the algebra of clopen subsets is countably complete. Unless stated otherwise the spaces we are considering are all zero-dimensional.

Proposition 6.2. *Let X be the topological sum of countably many compact spaces that are not basically disconnected. If its remainder X^* is $(\aleph_1, \aleph_0)$-ideal then $\mathfrak{d} = \aleph_1$.*

Proof. Write $X = \bigoplus_{n\in\omega} X_n$ and fix for each n an infinite family $\{a(n,m)\colon m \in \omega\}$ of pairwise disjoint clopen sets of X_n so that D_n, the closure of their union, is not open.

Assume that M is an $\aleph_0$-covering elementary substructure of some $H(\theta)$ of size $\aleph_1$ that contains X and the family $\{a(n,m)\colon n,m \in \omega\}$. We show that $M \cap \mathbb{N}^{\mathbb{N}}$ is cofinal in $\mathbb{N}^{\mathbb{N}}$. Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function not in M. We find $g \in M$ such that $f \leqslant g$.

Let $b = \bigcup\{a(n,m)\colon m \leqslant f(n)\}$; observe that b is also not in M. We take a countable subfamily C of $M \cap \mathrm{CO}(X)$ that is cofinal in $b^{\perp} \cap M$—this means that $c \cap b$ is compact for all $c \in C$ and that whenever $d \in M$ and $d \cap b$ is compact there is $c \in C$ such that the difference $d \setminus c$ is compact.

There are two cases to consider. If there is a $c \in C$ such that the set $I_c = \{n\colon (\exists m)\ (c \cap a(n,m) \neq \emptyset)\}$ is infinite then we are done. Indeed, define $h \in M$ by $h(n) = \min\{m\colon c \cap a(n^+,m) \neq \emptyset\}$, where $n^+ = \min\{l \geqslant n\colon l \in I_c\}$. Because $c \cap b$ is compact there is an l such that $c \cap a(n,m) = \emptyset$ whenever $n \leqslant l$ and $m \leqslant f(n)$. It follows that for $n \geqslant l$ we have $h(n) = h(n^+) > f(n^+) > f(n)$. Now define g by $g(n) = \max\{h(n), f(n)\}$; this g belongs to M because it is a finite modification of h and it is as required.

In the other case, where I_c is finite for all c, we may assume $C \in M$: indeed, take a countable $D \in M$ with $C \subseteq D$ and replace C by the set of elements d of D for which I_d is finite. By subtracting a compact part from each c we can also assume that every c is disjoint from every $a(n,m)$.

But now from an enumeration $\{c_n\colon n \in \omega)$ of C (that is in M) we define the clopen set

$$c = \bigcup\bigl\{X_n \cap (c_0 \cup \cdots \cup c_n)\colon n \in \omega\bigr\}.$$

It follows that c is in $M \cap b^{\perp}$. Now for each n, $D_n \cap c$ is empty, but D_n is not equal to $X_n \setminus c$ since D_n is not open. Therefore, there is some $d \in M \cap b^{\perp}$ such that $c \subseteq d$ and, for each n, $d \cap X_n \setminus c$ is not empty. It follows that $d \setminus c_k$ is not compact for any k and so $\{c_k\colon k \in \omega\}$ is not a generating set for $b^{\perp} \cap M$. Therefore this case does not occur. □

Theorem 6.3. $(\neg\mathrm{CH})$ *If $X = \bigoplus_{n\in\omega} X_n$ is the topological sum of countably many compact spaces that are not basically disconnected then the remainder of X is not Cohen–Parovičenko.*

Proof. Choose $x_n \in X_n$ for all n and observe that $D = \mathrm{cl}\{x_n\colon n \in \omega\} \cap X^*$ is homeomorphic to $\mathbb{N}^*$. The map that send X_n to the point x_n induces an open retraction from X^* onto D. It follows that $\mathfrak{d} \geqslant \mathfrak{d}_2(X^*)$, so if $\mathrm{CO}(X^*)$ is $(\aleph_1, \aleph_0)$-ideal then, by Proposition 6.2, we get $\mathfrak{d}_2(X^*) = \aleph_1 < \mathfrak{c}$. $\square$

Remark 6.4. Clearly it follows from the previous result that if $\mathbb{N}^*$ is Cohen–Parovičenko and CH fails then $(\omega \times (\omega+1))^*$ is not homeomorphic to ω^*. Using this fact and tracking the location of both clopen and nowhere dense P-set copies of $\mathbb{N}^*$ in their remainders, one can easily show that, in addition, $\omega \times (\omega+1)$ and $\omega \times (\omega^2+1)$ do not have homeomorphic remainders either.

It is also worth mentioning the following result since it has already found applications in Functional Analysis, see [10].

Corollary 6.5. $(\neg\mathrm{CH})$ *If $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is Cohen–Parovičenko and C is a non-compact cozero set in $\mathbb{N}^*$, then the closure of C is not a retract of $\mathbb{N}^*$.*

Proof. Let C be a non-compact cozero subset of $\mathbb{N}^*$. It follows that C is a countable union of compact open subsets of $\mathbb{N}^*$ and, as is well known, that the closure of C is just its Čech–Stone compactification. Now if the closure were a retract of $\mathbb{N}^*$, then its clopen algebra would be an $(\aleph_1, \aleph_0)$-algebra. The boundary of C, which is homeomorphic to $\beta C \setminus C$, is a G_δ-set in the closure of C; hence its clopen algebra is also an $(\aleph_1, \aleph_0)$-algebra by Lemma 5.11.

However, no clopen subset of $\mathbb{N}^*$ is basically disconnected so by Proposition 6.2 we have $\mathfrak{d} = \aleph_1$. But we assumed that $\mathfrak{m}_c$ and hence $\mathfrak{d}$ was equal to $\mathfrak{c}$. $\square$

With the previous results in mind it is tempting to hope that for σ-compact locally compact X and Y, if X^* and Y^* were homeomorphic then X and Y would be homeomorphic-modulo-compact-sets in some sense. For example, we do not know if $(\omega \times 2^\omega)^*$ and $(\omega \times 2^{\omega_1})^*$ are homeomorphic in the Cohen model.

We do however know of other spaces whose remainder is Cohen–Parovičenko. The proof of this fact is a rather interesting use of the basic results we have developed about Cohen–Parovičenko algebras. Recall that the Gleason cover or absolute of a compact space X is denoted by $E(X)$ and that $E(X)$ is just the Stone space of the complete Boolean algebra of regular open subsets of X. We write E_κ for $\omega \times E(2^\kappa)$.

Lemma 6.6. *Let $X = \bigoplus_{n\in\omega} X_n$ be the topological sum of basically disconnected compact spaces. Then $\mathfrak{d}_2(X^*) = \mathfrak{d}$.*

Proof. Lemma 6.1 gives us $\mathfrak{d} \geqslant \mathfrak{d}_2(X^*)$. To prove the other inequality we take an ideal in $\mathrm{CO}(X^*)$ that is co-generated by a strictly increasing sequence. Translating this into $\mathrm{CO}(X)$ we get an increasing sequence $\langle C_k\colon k \in \omega\rangle$ of clopen sets in X such that for all k the difference $C_{k+1}\setminus C_k$ is not compact (for convenience we assume $C_0 = \emptyset$), and the ideal $\mathcal{I}$ in $\mathrm{CO}(X)$ consisting of those sets D for which every intersection $D \cap C_k$ is compact.

For every $n, k \in \omega$ put $a(n,k) = X_n \cap (C_{k+1}\setminus C_k)$. This transforms the C_k into an increasing sequence $\langle c_k\colon k \in \omega\rangle$ of subsets of the countable set $A = \{a(n,k)\colon n, k \in \omega\}$, where $c_k = \{a(n,l)\colon n \in \omega,\ l < k\}$. To every $D \in \mathcal{I}$ corresponds the set $x_D = \{a \in A\colon D \cap a \neq \emptyset\}$; observe that $x_D \cap c_k$ is finite for each k. Because each X_n is basically disconnected the sets $D_n = \mathrm{cl}\bigcup_k a(n,k)$ are clopen (maybe empty); we put $Y = X \setminus \bigcup_n D_n$.

Let $\mathcal{J}$ be a subfamily of $\mathcal{I}$ of size less than $\mathfrak{d}$ and consisting of non-compact sets. Fix an infinite subset d of A such that $d \cap c_k$ is finite for all k and such that $d \cap x_D$ is infinite whenever $D \in \mathcal{J}$ and x_D is infinite. Finally put $C = Y \cup \bigcup d$. Clearly $C \cap C_k$ is compact for every k. If $D \in \mathcal{J}$ and x_D is finite then $D \cap Y$ is not compact; if x_D is infinite then $D \cap \bigcup d$ is not compact. □

Remark 6.7. A similar result does not hold for $\mathfrak{r}$. Indeed, consider the space E_κ; a clopen set in its remainder E_κ^* is determined by a clopen set of E_κ itself. In turn a clopen subset of E_κ is determined by a regular open subset of $\omega \times 2^\kappa$ and it is well known that such a regular open set depends on at most countably many coordinates. Thus, if $\mathcal{C}$ is a family of fewer than κ many clopen sets in E_κ^* then we can find an $\alpha \in \kappa$ such that no element of $\mathcal{C}$ depends on α. But this means that the clopen set $\omega \times \pi_\alpha^{\leftarrow}(0)$ (or rather the clopen subset of E_κ^* determined by it) reaps the family $\mathcal{C}$. We deduce that $\mathfrak{r}(E_\kappa^*) \geqslant \kappa$ and hence that, for example, $\mathfrak{r}(E_\mathfrak{c}^*) > \mathfrak{r}$ in models where $\mathfrak{c} > \mathfrak{r}$.

Theorem 6.8. *For each cardinal $\kappa \leqslant \mathfrak{c}$, the remainder of E_κ is Cohen–Parovičenko iff $\mathbb{N}^*$ is Cohen–Parovičenko.*

Proof. We start out by observing two partial equivalences.

Claim 1. $\mathfrak{d} = \mathfrak{c}$ *iff* $\mathfrak{d}_2(E_\kappa^*) = \mathfrak{c}$.

Proof of Claim 1. By Lemma 6.6 we know that $\mathfrak{d}_2(E_\kappa^*) = \mathfrak{d}$ for all κ.

Claim 2. *The algebra* $\mathrm{CO}(E_\kappa^*)$ *is* $(*, \aleph_0)$*-ideal iff* $\mathcal{P}(\mathbb{N})/\mathit{fin}$ *is.*

Proof of Claim 2. This follows by applying Theorem 5.15 twice. First: $\mathbb{N}^*$ is easily seen to be a retract of E_κ^*, so $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is $(*, \aleph_0)$-ideal if $\mathrm{CO}(E_\kappa^*)$ is. Second: βE_κ is a separable extremally disconnected compact space and hence can be embedded as retract in $\mathbb{N}^*$, so that $\mathrm{CO}(\beta E_\kappa^*)$ is $(*, \aleph_0)$-ideal if $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is and, by Corollary 5.12, so is the clopen algebra of E_κ^*.

We would be done if we could also prove $\mathfrak{r}(E_\kappa^*) = \mathfrak{r}$ but by Remark 6.7 we know that this cannot be done. We circumvent this difficulty by showing that $\mathfrak{r} \geqslant \mathfrak{d}$ if $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is

$(*, \aleph_0)$-ideal. This will follow from the following technical lemma, which is in the spirit of Proposition 2.3 of [13], whose content was explained just before Theorem 5.6. □

Lemma 6.9. *If $\kappa < \mathfrak{d}$ and $\mathcal{P}(\mathbb{N})/fin$ is $(\kappa, \aleph_0)$-ideal then also $\kappa < \mathfrak{r}$.*

Proof. It suffices to show that if $M \prec H(\theta)$ is $\aleph_0$-covering, of size κ and such that $M \cap \mathcal{P}(\mathbb{N})$ is $\aleph_0$-ideal in $\mathcal{P}(\mathbb{N})$ then there is an $r \in \mathcal{P}(\mathbb{N})$ that reaps $M \cap [\mathbb{N}]^{\aleph_0}$. By van Douwen's characterization of $\mathfrak{d}$ (see Definition 4.2) there is $f \in {}^{\mathbb{N}}\mathbb{N}$ such that for every $x \in M \cap [\mathbb{N}]^{\aleph_0}$ and every $g \in M \cap {}^{\mathbb{N}}\mathbb{N}$ there is an $n \in x$ such that $f(n) \geqslant g(n)$. Fix a countable subset C of $M \cap {}^{\mathbb{N}}\mathbb{N}$ such that for every subset a of $\mathbb{N} \times \mathbb{N}$ with $a \subseteq L_f$ there is $c \in C$ such that $a \in L_c$. As we can assume $C \in M$ and because M knows that C is countable we can find $g \in M \cap {}^{\mathbb{N}}\mathbb{N}$ such that $c <^* g$ for all $c \in C$. We claim that $r = \{n\colon f(n) \geqslant g(n)\}$ is as required.

Now let $x \in M \cap [\mathbb{N}]^{\aleph_0}$; the choice of f implies that $r \cap x$ is infinite. To show that $r' \cap x$ is infinite consider $a = L_g \cap (x \times \omega)$. Clearly there is no $c \in C$ with $a \subseteq^* L_c$, hence $a \backslash L_f$ is infinite; this gives infinitely many n with $g(n) > f(n)$. □

Remark 6.10. Many of the foregoing consequences of $\mathbb{N}^*$ being Cohen–Parovičenko do need the assumption of $\neg$CH. For example, it is shown in [7] that a homeomorphism between nowhere dense P-set subsets of $\mathbb{N}^*$ can be lifted to a homeomorphism on $\mathbb{N}^*$. In addition, Steprāns [23] proves that all P-points can be taken to one another by autohomeomorphisms of $\mathbb{N}^*$ in the Cohen model (and it appears that only the assumption that $\mathfrak{m}_c = \aleph_2 = \mathfrak{c}$ is used). However we can provide the following elegant contrasting result.

Proposition 6.11. *If $\mathfrak{c} = \aleph_2$ and if $\mathbb{N}^*$ is Cohen–Parovičenko then there are two P-sets in $\mathbb{N}^*$, of character $\aleph_1$ and $\aleph_2$ respectively, that are both homeomorphic to $\mathbb{N}^*$.*

Proof. Using Theorem 6.8 we see that $E^*_{\mathfrak{c}}$ is Cohen–Parovičenko. We may therefore apply Theorem 5.5 to deduce that $\mathbb{N}^*$ and $E^*_{\mathfrak{c}}$ are homeomorphic. Now fix one point x in $E(2^{\mathfrak{c}})$; the set $(\omega \times \{x\})^*$ is a P-set of character $\mathfrak{c}$ in $E^*_{\mathfrak{c}}$ and clearly homeomorphic to $\mathbb{N}^*$.

To get a P-set of character $\aleph_1$ we take a strictly decreasing chain $\langle a_\alpha\colon \alpha < \omega_1\rangle$ of clopen sets in $\mathbb{N}^*$ whose intersection A is nowhere dense in $\mathbb{N}^*$—see Remark 2.11 for the construction. Clearly then A is a P-set of character $\aleph_1$. The ideal $\mathcal{I}$ generated by the family $\{a'_\alpha\colon \alpha < \omega_1\}$ is $\aleph_1$-generated, so by Corollary 5.14 the algebra $(\mathcal{P}(\mathbb{N})/fin)/\mathcal{I}$ is Cohen–Parovičenko and hence isomorphic to $\mathcal{P}(\mathbb{N})/fin$. Its Stone space is A, which consequently is homeomorphic to $\mathbb{N}^*$. □

7. Problems

Other reals. The Cohen model is probably the most intensively investigated model of $\neg$CH of all; this may explain our success in extracting key features of $\mathcal{P}(\mathbb{N})/fin$ and $\mathbb{N}^*$ in that model. It would be of great interest if a similar thing could be done for other familiar models of $\neg$CH.

The Laver and Sacks (also side-by-side) models are particular favourites of the authors but the Random real model seems the most likely candidate for a successful investigation.

Characterizing $\mathcal{P}(\mathbb{N})/fin$. Theorems 2.13 and 5.5 lead one to hope that there is a characterization of $\mathcal{P}(\mathbb{N})/fin$ in any Cohen model. As first steps on the way to such a result we ask the following questions.

Question 2. Is, in the $\aleph_3$-Cohen model, $\mathcal{P}(\mathbb{N})/fin$ the unique Cohen–Parovičenko algebra?

Or, more generally:

Question 3. If $\mathfrak{c} = \aleph_3$ are then all Cohen–Parovičenko algebras of cardinality $\mathfrak{c}$ isomorphic?

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