Positivity 10 (2006), 607–611

1385-1292/030607-5, *published online* May 24, 2006
DOI 10.1007/s11117-005-0031-0

Positivity

A Connected F-Space

KLAAS PIETER HART
Faculty of Electrical Engineering, Mathematics, and Computer Science, TU Delft, Postbus 5031, 2600 GA Delft, The Netherlands. E-mail: K.P.Hart@EWI.TUDelft.NL

Received 14 April 2004; accepted 8 June 2004

Abstract. We present an example of a compact connected F-space with a continuous real-valued function f for which the set $\Omega_f = \bigcup\{\mathrm{Int}\, f^{\leftarrow}(x) : x \in \mathbb{R}\}$ is not dense. This indirectly answers a question from Abramovich and Kitover in the negative.

Mathematics Subject Classification 2000: Primary: 54G20, Secondary: 46A40, 54D30, 54F15, 54G05

Key words: Continuum, d-basis, d-independence, F-space.

1. Introduction

The purpose of this note is to give a positive answer to Problem 4 from Abramovich-Kitover [1]. The problem asks whether there are a compact and connected F-space K and a continuous real-valued function f on K such that the set Ω_f is not dense in K, where $\Omega_f = \bigcup\{\mathrm{Int}\, f^{\leftarrow}(x) : x \in \mathbb{R}\}$. If K is such a space then the vector lattice $C(K)$ has a maximal d-independent system that is not a d-base, which answers Problem 1 from the same paper in the negative.

As defined in Abramovich-Kitover [1] a *d-independent system* in a vector lattice X is a subset D with the property that for every band B in X, for every finite subset F of D and every choice $\{c_d : d \in F\}$ of nonzero scalars the condition $\sum_{d\in F} c_d d \perp B$ implies $d \perp B$ for all $d \in F$. A d-independent system D is a *d-basis* if for every $x \in X$ one can find a full system $\mathcal{B}$ of pairwise disjoint bands and a subset $\{y_B : B \in \mathcal{B}\}$ of X such that for each B the element y_B is a linear combination of members of D and $x - y_B \perp B$.

In topological terms a d-independent system in $C(K)$ is a subset D such that for every nonempty open subset O the family of nonzero members of $\{d \restriction O: d \in D\}$ is linearly independent. The d-independent set D is a d-basis if for each $g \in C(K)$ there is a pairwise disjoint family $\mathcal{O}$ of open sets with a dense union and such that for every $g \in C(K)$ and every $O \in \mathcal{O}$ the restriction $g \restriction O$ is a linear combination of finitely members of $\{d \restriction O: d \in D\}$.

As observed in Abramovich-Kitover [1] for our example K the set $\{1\}$, consisting of just the constant function with value 1, is maximally d-indepent in $C(K)$. Indeed, if g is not constant then its image $g[K]$ is a nontrivial interval; we let t be its mid-point. Because K is an F-space the closed sets $\operatorname{cl} g^{\leftarrow}\big[(-\infty, t)\big]$ and $\operatorname{cl} g^{\leftarrow}\big[(t, \infty)\big]$ are disjoint and because K is connected they do not cover K. The nonempty open set $\operatorname{Int} g^{\leftarrow}(t)$ now witnesses that $\{1, g\}$ is not d-independent. The continuous function f, on the other hand, witnesses that $\{1\}$ is not a d-basis, for clearly any 'd-linear combination' g of $\{1\}$ must have its set Ω_g dense in K.

2. The Example

Let S be the unit square, i.e., $S = [0, 1]^2$. We consider the product $\mathbf{S} = \omega \times S$, its Čech–Stone compactification $\beta\mathbf{S}$ and the extension $\beta\pi$ of the map $\pi : \mathbf{S} \to \omega$, defined by $\pi(n, x) = n$.

For each free ultrafilter $u \in \beta\omega \setminus \omega$ the fiber $S_u = \beta\pi^{\leftarrow}(u)$ is a continuum–see, e.g., Hart [2]. As it is a closed subset of the Čech–Stone remainder $\mathbf{S}^*$ it is also a compact F-space.

The function $f : \mathbf{S} \to [0, 1]$, defined by $f(n, x, y) = x$ is clearly continuous; we write f_u for the restriction of βf to S_u. We shall find a continuum K in S_u such that $g = f_u \restriction K$ is as required, i.e., Ω_g is not dense in K.

We need to describe the boundaries of the fibers of f. We define $L_t = f_u^{\leftarrow}(t) \cap \operatorname{cl} f_u^{\leftarrow}\big[[0, t)\big]$ and $R_t = f_u^{\leftarrow}(t) \cap \operatorname{cl} f_u^{\leftarrow}\big[(t, 1]\big]$; note that $L_0 = R_1 = \emptyset$.

LEMMA 2.1. *For each $t \in (0, 1)$ the sets L_t and R_t are exactly the components of the boundary* $\operatorname{Fr} f_u^{\leftarrow}(t)$ *of* $f_u^{\leftarrow}(t)$.

Proof. Because S_u is an F-space the closed sets L_t and R_t are disjoint; they cover $\operatorname{Fr} f_u^{\leftarrow}(t)$ and, because S_u is connected, both are nonempty. This shows that $\operatorname{Fr} f_u^{\leftarrow}(t)$ has at least two components.

To finish we show that L_t and R_t are connected. For this we first observe that the 'rectangle' $P_{s,r} = S_u \cap \operatorname{cl}\big(\omega \times [s, r] \times [0, 1]\big)$ is connected whenever $s < r$. This in turn implies that $L_{s,t} = \operatorname{cl} \bigcup_{s<r<t} P_{s,r}$ is connected whenever $s < t$. It is readily verified that $L_t = \bigcap_{s<t} L_{s,t}$, hence L_t is connected as the intersection of a chain of continua. By symmetry R_t is also connected. □

This argument also shows that $R_0 = \operatorname{Fr} f_u^{\leftarrow}(0)$ and $L_1 = \operatorname{Fr} f_u^{\leftarrow}(1)$ are connected.

We need some more notation. We denote by B_u the intersection of S_u with the closure, in $\beta\mathbf{S}$, of $\omega \times [0, 1] \times \{0\}$—the bottom line of S_u—and likewise the top line T_u is $S_u \cap \operatorname{cl}_{\beta\mathbf{S}}(\omega \times [0, 1] \times \{1\})$. The continuum K will be defined as the union of the bottom line of S_u and a family of vertical continua, each of which meet both the bottom and top lines.

To define this family we define sequences $\langle X_\alpha \rangle_\alpha$ and $\langle f_\alpha \rangle_\alpha$ of closed sets and functions, respectively, by recursion. To begin let $X_0 = S_u$. Given X_α put $f_\alpha = f_u \restriction X_\alpha$ and define $X_{\alpha+1} = X_\alpha \setminus \bigcup_t \operatorname{Int}_\alpha f_\alpha^{\leftarrow}(t)$, where $\operatorname{Int}_\alpha$ is the interior operator in X_α. If α is a limit we just let $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$.

LEMMA 2.2. *For every α and every t the intersections $X_\alpha \cap L_t$ and $X_\alpha \cap R_t$ are nonempty*

Proof. The proof is by induction on α.

The statement is clearly true for $\alpha = 0$ and the case $\alpha = 1$ is covered by Lemma 2.1, whose proof also establishes the successor step in the induction. Indeed, to show that $X_{\alpha+1} \cap L_t \neq \emptyset$ we note that, by the inductive assumption we know that $P_{s,r} \cap X_\alpha$ meets L_q and R_q, whenever $s < q < r$. Therefore, $L_{s,t} \cap X_\alpha \neq \emptyset$ for all $s < t$; using compactness we find that $L_t \cap X_{\alpha+1} = \bigcap_{s<t}(L_{s,t} \cap X_\alpha)$ is nonempty.

The case of limit α follows using compactness as well. □

LEMMA 2.3. *Every component of X_α meets both B_u and T_u.*

Proof. This is clear when $\alpha = 0$ and as in the previous lemma we draw inspiration from the proof of Lemma 2.1 for the argument in the successor step. Observe first that a component of $X_{\alpha+1}$ is necessarily a subset of some L_t or R_t: these sets are the components of X_1.

Let C be a component of L_t and let O be an arbitrary clopen neighbourhood of C in $L_t \cap X_{\alpha+1}$; choose open sets U and V in S_u with disjoint closures such that $O \subseteq U$ and $(L_t \cap X_{\alpha+1}) \setminus O \subseteq V$. There is an s such that $L_{s,t} \cap X_\alpha \subseteq U \cup V$. Choose $r \in (s,t)$ such that some component, D, of $X_\alpha \cap (L_r \cup R_r)$ meets U; then $D \subseteq U$ and it follows that U intersects both B_u and T_u. Because O and U were arbitrary it follows that C must meet B_u and T_u as well.

In case α is a limit and C a component we have $C = \bigcap_{\beta<\alpha} C_\beta$, where C_β is the component of X_β that contains C; the C_β's form a chain and all of them intersect B_u and T_u and hence by compactness so does C. □

There will be a minimal ordinal δ such that $X_\delta = X_{\delta+1}$ (some information on δ will be given in Section 3). This means that $\mathrm{Int}_\delta\, f_\delta^{\leftarrow}(t) = \emptyset$ for all t.

Our continuum K is the union of B_u and X_δ. Because all components of X_δ meet B_u we know that K is indeed connected. Because each component meets T_u we know that K reaches all the way up to T_u; by the choice of δ we get that $\mathrm{Int}_K\, g^{\leftarrow}(t) \subseteq B_u$ for all t. Thus $\Omega_g \subseteq B_u$ and the latter set is certainly not dense in K.

3. A Remark and a Question

The first (and erroneous) version of K was simply $B_u \cup \bigcup_{0<t\leqslant 1} R_t \cup \bigcup_{0\leqslant t<1} L_t$. After I realized that the restriction of f to this subspace did not provide an example it became clear that the procedure of removing interiors of fibers had to be iterated, which lead to the sequence $\langle X_\alpha \rangle_\alpha$. We can provide some information on the ordinal δ at which the sequence becomes constant.

PROPOSITION 3.1. $\delta < \mathfrak{c}^+$

Proof. Let $\mathcal{B}$ be a base for S_u of cardinality $\mathfrak{c}$. For every $\alpha < \delta$ there is a $B_\alpha \in \mathcal{B}$ such that $\emptyset \neq B_\alpha \cap X_\alpha \subseteq X_\alpha \setminus X_{\alpha+1}$. Clearly $\alpha \mapsto B_\alpha$ is one-to-one, which establishes that $|\delta| \leqslant \mathfrak{c}$.

□

The F-space property implies that δ cannot be a successor ordinal, nor an ordinal of countable cofinality.

LEMMA 3.1. *If $\alpha < \delta$ then $X_\alpha \setminus X_{\alpha+1}$ meets every L_t and every R_t.*

Proof. This is basically a consequence of the homogeneity of the unit interval. If $h : [0, 1] \to [0, 1]$ is a homeomorphism then it induces an autohomeomorphism h_u of S_u via the map $(n, x, y) \mapsto (n, h(x), y)$ from $\mathbf{S}$ to itself. The map h_u simply permutes the fibers $f^{\leftarrow}(t)$ and it is relatively straightforward to show by induction that $h_u[X_\alpha] = X_\alpha$ for all α. There are enough maps h to ensure that once $X_\alpha \setminus X_{\alpha+1}$ meets one L_t (or one R_t) it meets all L_s and all R_s. □

PROPOSITION 3.2. *δ is not a successor ordinal.*

Proof. Let $\alpha < \delta$, we show that $\alpha + 1 < \delta$. Fix $t \in (0, 1)$ and let $\langle t_n \rangle_n$ be a sequence in $[0, 1]$ that converges to t from above. By Lemma 2.2 we can pick $x_n \in L_{t_n} \cap X_\alpha \setminus X_{\alpha+1}$ for each n.

Clearly every point in the closure of $\{x_n\}_n$ belongs to $X_{\alpha+1} \cap R_t$; we show that none belong to $X_{\alpha+2}$. To see this observe that the F_σ-sets $F = \{x_n\}_n$ and $G = f^{\leftarrow}\big[(t, 1]\big]$ are *separated* in S_u, i.e., $\operatorname{cl} F \cap G = \emptyset = F \cap \operatorname{cl} G$. Using normality in the form of Urysohn's lemma one can find a continuous function $h : S_u \to [-1, 1]$ such that $h[F] \subseteq [-1, 0)$ and $h[G] \subseteq (0, 1]$. But now the F-space property applies to show that $\operatorname{cl} F \cap \operatorname{cl} G = \emptyset$. □

In a similar way we can prove the following.

PROPOSITION 3.3. *The ordinal δ has uncountable cofinality.*

Proof. We choose an increasing sequence $\langle \alpha_n \rangle_n$ of ordinals below δ; we show that $\lim_n \alpha_n < \delta$.

As in the previous proof we fix $t \in (0, 1)$ and a sequence $\langle t_n \rangle_n$ converging to t from above. As before we choose $x_n \in L_{t_n} \cap X_{\alpha_n} \setminus X_{\alpha_n+1}$ for all n.

As in the previous proof the F-space property now ensures that every point in the closure of $\{x_n\}_n$ belongs to $X_\alpha \setminus X_{\alpha+1}$. □

We deduce that δ must be at least ω_1 but the following question remains.

QUESTION 1. *What is the exact value of* δ*?*

References

1. Abramovich, Y.A. and Kitover, A.K.: d-Independence and d-bases, Positivity, **7** (2003), 95–97.
2. Hart, K.P.: The Čech-Stone compactification of the real line, Recent Progress in General Topology, edited by M. Hušek, and J. van Mill, North-Holland, Amsterdam, 1992, pp. 317–352, MR 95g:54004.