ON HEREDITARILY INDECOMPOSABLE COMPACTA AND FACTORIZATION OF MAPS

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ABSTRACT. We employ a dual version of the Löwenheim-Skolem theorem to obtain a factorization theorem for maps with hereditarily indecomposable fibers. This enables us to obtain universal hereditarily indecomposable compact spaces as well as hereditarily indecomposable compactifications of any prescribed weight and dimension. We also reprove theorem of Maćkoviak on the existence of universal hereditarily indecomposable continua.

1. INTRODUCTION

All spaces are assumed to be normal. By a map we mean a continuous function. We say that a compactum X is *hereditarily indecomposable* if for every two intersecting continua in X one is contained in the other. The main result of this note is the following theorem.

Theorem 1.1. *Let $f : X \to Y$ be a perfect map with hereditarily indecomposable fibers from a separable metrizable space X onto a zero-dimensional separable metrizable space Y. Then there are a hereditarily indecomposable metrizable compactification $X^\star$ of X with $\dim X^* = \dim X$ and a zero-dimensional metrizable compactification Y^* of Y such that f can be extended to a map $f^* : X^* \to Y^*$.*

Let us note that this result, combined with a pseudosuspension method, yields a theorem of Maćkowiak [10] on the existence of universal n-dimensional hereditarily indecomposable continua. This theorem was obtained by Maćkowiak by a quite

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different method based on a subtle use of inverse limits. We comment on this in Corollary 4.1.

Rather unexpectedly, our proof uses, in an essential way, large nonmetrizable compactifications and a considerable strengthening of Mardešić's Factorization Theorem (see [3, Theorem 3.4.1]). This strengthening is a dual version of the Löwenheim-Skolem theorem from model theory; it appears as Theorem 3.1 in [2] and it was put to good use in [4] and [12]. In Section 2 we explain some general facts concerning this technique and in section 3 we show how our theorem follows from these results. Among other consequences of this technique is the following theorem, proved in section 3.

Theorem 1.2. *For every cardinal τ and $n \in \{0, 1, \ldots, \infty\}$ there exists a hereditarily indecomposable compactum $X(n, \tau)$ of weight τ and dimension n that contains a copy of every hereditarily indecomposable compactum of weight not more than τ and dimension at most n.*

The following property of a space X was formulated by Krasinkiewicz and Minc [6]:

Property (KM). For every two disjoint closed sets C and D in X and disjoint open sets U and V in X with $C \subset U$ and $D \subset V$ there exist closed sets X_0, X_1 and X_2 in X such that $X = X_0 \cup X_1 \cup X_2$, $C \subset X_0$, $D \subset X_2$, $X_0 \cap X_1 \subset V$, $X_1 \cap X_2 \subset U$ and $X_0 \cap X_2 = \emptyset$.

To avoid having to write down the six conditions each time we shall call a triple $\langle X_0, X_1, X_2 \rangle$ a *fold* of X for the quadruple $\langle C, D, U, V \rangle$.

Theorem 1.3 ([6]). *A compact space is hereditarily indecomposable if and only if it has Property (KM).*

2. A FACTORIZATION METHOD

The factorization method alluded to in the Introduction is based on a mix of Model Theory and Set-Theoretic Topology. It works best in the realm of compact Hausdorff spaces, as will become clear shortly.

The first ingredient is Wallman's representation theorem, [13], for distributive lattices: if L is such a lattice then the set wL of ultrafilters on L carries a natural compact T_1-topology. This topology has the family $\{\bar{a} : a \in L\}$ as a base for the closed sets, where $\bar{a} = \{u \in wL : a \in u\}$.

If X is compact and T_1 and L is the family of closed subsets of X, with union and intersection as its lattice operations then $x \mapsto u_x = \{a \in L : x \in a\}$ is a

homeomorphism from X onto wL; this remains true if L is a base for the closed sets of X that is closed under unions and intersections. See, e.g., [1] for a short introduction to Wallman representations.

For a normal space X one can obtain the Čech-Stone compactification, βX, as the Wallman representation of the lattice of closed sets of X. This is the key to the next theorem.

Theorem 2.1. *If X has Property (KM) then so does its Čech-Stone compactification βX and, in particular, βX is hereditarily indecomposable.*

PROOF. To begin: it should be clear that Property (KM) can be (re)formulated in terms of closed sets only and that it is a finitary lattice-theoretic property; one can express it as an implication involving seven variables. Thus if X has Property (KM) then the canonical base, $\mathcal{B}$, for the closed sets of βX satisfies this implication. This does not automatically imply that βX has Property (KM), because that means that the full family of closed sets of βX satisfies the lattice-theoretic formula. However, in the present case one can start with arbitrary C, D, U and V and use compactness and the fact that $\mathcal{B}$ is closed under finite unions and intersections to find C', D', U' and V' such that $C \subseteq C' \subseteq U' \subseteq U$ and $D \subseteq D' \subseteq V' \subseteq V$, and such that C', D', $\beta X \setminus U'$ and $\beta X \setminus V'$ belong to $\mathcal{B}$. One can then find a fold $\langle X_0, X_1, X_2 \rangle$ for $\langle C', D', U', V' \rangle$ in $\mathcal{B}$ and this will also be a fold for $\langle C, D, U, V \rangle$. $\square$

The second ingredient is the use of notions from Model Theory, especially elementary substructures and the Löwenheim-Skolem theorem. In the context of lattices elementarity is perhaps best explained in terms of solutions to equations. One can interpret Property (KM) as saying that certain equations should have solutions: the quadruple $\langle C, D, U, V \rangle$ determines six equations and a fold $\langle X_0, X_1, X_2 \rangle$ is a solution to this system.

One calls M an elementary sublattice of L if every lattice-theoretic equation with constants from M that has a solution in L also has a solution in M.

To illustrate its use we prove the following lemma.

Lemma 2.2. *Assume X is a hereditarily indecomposable compact space and let L be an elementary sublattice of the lattice of closed subsets of X. Then wL is also hereditarily indecomposable.*

PROOF. By elementarity the lattice L satisfies Property (KM): if C, D, $X \setminus U$ and $X \setminus V$ belong to L then there is a fold $\langle X_0, X_1, X_2 \rangle$ in the full family of closed sets, *hence* there is also such a fold in L.

Next, in wL the same argument as in the proof of Theorem 2.1 applies: an arbitrary quadruple can be expanded to a quadruple from the base. $\qquad\square$

The Löwenheim-Skolem Theorem provides us with many elementary substructures: given a lattice L and some subset A of L one can construct an elementary sublattice L_A of L that contains A and whose cardinality is at most $|A| \times \aleph_0$.

Theorem 2.3 ([2, 12, 4]). *Let $f : X \to Y$ be a continuous surjection from a hereditarily indecomposable compact space onto a compact space. Then there are a compact space Z and continuous maps $g : X \to Z$ and $h : Z \to Y$ such that Z is hereditarily indecomposable,* $\dim Z = \dim X$, $w(Z) = w(Y)$ *and $f = h \circ g$.*

PROOF. Let $\mathcal{B}$ be a base for the closed sets of Y of cardinality $w(Y)$. Via $B \mapsto f^{-1}[B]$ we can identify $\mathcal{B}$ with a sublattice of the lattice $\mathcal{D}$ of closed subsets of X.

Apply the Löwenheim-Skolem Theorem to find an elementary sublattice $\mathcal{C}$ of $\mathcal{D}$ that contains $\mathcal{B}$ and has the same (infinite) cardinality as $\mathcal{B}$; we let $Z = w\mathcal{C}$. The two inclusions $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{D}$ induce continuous surjections $g : X \to Z$ and $h : Z \to Y$ that, as one readily shows, satisfy $f = h \circ g$. By Lemma 2.2 the space Z is hereditarily indecomposable. The same argument shows that $\dim Z = \dim X$: one can use, for example, the Theorem on Partitions, [3, Theorem 1.7.9], to turn the statement $\dim X \leqslant n$ into an equation Φ_n. By elementarity $\mathcal{C}$ and $\mathcal{D}$ satisfy Φ_n for exactly the same values of n. The expansion trick applies in this case as well so that $\dim Z \leqslant n$ for exactly the same values of n for which $\mathcal{C}$ satisfies Φ_n. $\qquad\square$

We refer to [5] for basic information on Model Theory.

Remark. The thesis [12] contains a systematic study of properties that are preserved by continuous maps that are induced by elementary embeddings.

3. PROOFS OF THE MAIN RESULTS

We start with the following

Theorem 3.1. *Let $f : E \to F$ be a perfect mapping from a space E onto a strongly zero-dimensional paracompact space F such that for every $y \in F$ the fiber $f^{-1}(y)$ is hereditarily indecomposable. Then E has Property (KM).*

PROOF. Let C and D be disjoint closed subsets of E and let U and V disjoint open subsets of E around C and D respectively.

Let us fix $y \in F$. We shall find a (clopen) neighbourhood O_y of y and a fold of $f^{-1}[O_y]$ for $\langle C \cap f^{-1}[O_y], D \cap f^{-1}[O_y], U, V \rangle$. Since $f^{-1}(y)$ is compact and hereditarily indecomposable, by Theorem 1.3 it has Property (KM) and hence there exists a fold $\langle X_0, X_1, X_2 \rangle$ of $f^{-1}(y)$ for $\langle C \cap f^{-1}(y), D \cap f^{-1}(y), U, V \rangle$.

Apply [3, Theorem 3.1.1] to find a sequence $B = \langle W_0, W_1, W_2, O_U, U_V \rangle$ of open sets such that their closures form a swelling of the sequence $A = \langle C \cup X_0, X_1, D \cup X_2, X \setminus U, X \setminus V \rangle$, which means that each term of A is a subset of the corresponding term B and whenever I is such that $\bigcap_{i \in I} A_i = \emptyset$ then $\bigcap_{i \in I} \operatorname{cl} B_i = \emptyset$. Specifically this means that

(1) $f^{-1}(y) \subseteq W_0 \cup W_1 \cup W_2$;
(2) $\operatorname{cl} W_0 \cap \operatorname{cl} W_1 \subseteq V$;
(3) $\operatorname{cl} W_1 \cap \operatorname{cl} W_2 \subseteq U$;
(4) $\operatorname{cl} W_0 \cap \operatorname{cl} W_2 = \emptyset$.

As the map f is perfect and the space F is zero-dimensional we can find a clopen neighbourhood O_y of y such that $f^{-1}[O_y] \subseteq W_0 \cup W_1 \cup W_2$. It follows that $\langle \operatorname{cl} W_0, \operatorname{cl} W_1, \operatorname{cl} W_2 \rangle$ is a fold of $f^{-1}[O_y]$ for $\langle C \cap f^{-1}[O_y], D \cap f^{-1}[O_y], U, V \rangle$.

By strong zero-dimensionality and paracompactness we can find a disjoint clopen refinement $\mathcal{O}$ of $\{O_y : y \in F\}$; it is then a routine matter to combine the 'local' folds into one 'global' fold of E for $\langle C, D, U, V \rangle$. $\qquad\square$

We are now ready to prove the first main result.

PROOF OF THEOREM 1.1. To begin we construct a zero-dimensional compactification Y^* of Y, a compactification X_1 of X and a continuous extension $f_1 : X_1 \to Y^*$.

One way of doing this is by assuming that X is embedded in the Hilbert cube $I^{\aleph_0}$, that Y is embedded in the Cantor set $\{0,1\}^{\aleph_0}$ and then to identify X with the graph of f, i.e., X is identified with $G(f) = \{(x, f(x)) : x \in X\} \subseteq I^{\aleph_0} \times \{0,1\}^{\aleph_0}$ via $x \mapsto (x, f(x))$. After this identification f is simply $\pi_2 \restriction G(f)$, where π_2 is the projection onto the second factor of the product; we can then let $X_1 = \operatorname{cl} G(f)$ (in the product) and $Y^* = \operatorname{cl} Y$ (in the Cantor set), the desired extension f_1 of f then is $\pi_2 \restriction X_1$.

Next let $j : \beta X \to X_1$ be the natural map (the extension of the inclusion of X into X_1). By Theorem 3.1 X has Property (KM) so by Theorem 2.1 βX is hereditarily indecomposable. Apply Theorem 2.3 to obtain a factorization of j consisting of maps $g : \beta X \to X^*$ and $h : X^* \to X_1$ in which X^* is hereditarily indecomposable, second-countable and satisfies $\dim X^* = \dim \beta X = \dim X$. Then X^* is a metrizable compactification of X as $g \restriction X$ is a homeomorphism. It remains to set $f^* = f_1 \circ h$. $\qquad\square$

Let us note that since f is perfect and X^* is a compactification of X, the extension f^* satisfies $(f^*)^{-1}(y) = f^{-1}(y)$ for $y \in Y$.

To get universal hereditarily indecomposable compacta we use the factorization method again.

PROOF OF THEOREM 1.2. Let $\{X_s\}_{s\in S}$ be the family of all compact hereditarily indecomposable subspaces of the Tychonoff cube I^τ whose dimension is not larger than n, and let $i_s : X_s \to I^\tau$ be the inclusion. Let $X = \bigoplus_{s\in S} X_s$ be the free union of the X_s's and let $i : X \to I^\tau$ be defined by $i(x) = i_s(x)$ for $x \in X_s$. Let $f : \beta X \to I^\tau$ be the extension of i over βX. Obviously, X has Property (KM), so by Theorem 2.1 βX is hereditarily indecomposable. By Theorem 2.3 f can be factored as $h \circ g$, where $g : X \to Z$ and $h : Z \to Y$ and where Z is hereditarily indecomposable, $w(Z) \leqslant \tau$ and $\dim Z = \dim X$. We can take $X(n, \tau) = Z$. $\qquad\square$

4. COROLLARIES AND REMARKS

Let us note that as a corollary to either Theorem 1.1 or Theorem 1.2 one can obtain the following theorem of Maćkowiak [10].

Corollary 4.1. *For every $n \in \{1, 2, \dots \infty\}$ there exists a hereditarily indecomposable metric continuum Z_n of dimension n containing a copy of every hereditarily indecomposable metric continuum of dimension at most n.*

PROOF USING THEOREM 1.1. Let $\mathcal{P}$ be the subset of the hyperspace $2^{I^{\aleph_0}}$ of the Hilbert cube consisting of all hereditarily indecomposable continua of dimension n or less. Then $\mathcal{P}$ is a G_δ-subset of $2^{I^{\aleph_0}}$ (see [8, § 45, IV, Theorem 4 and § 48, V, Remark 5]). Therefore there is a continuous surjection $\varphi : Y \to \mathcal{P}$, where Y is the space of the irrationals. Then let X be the following subset of $I^{\aleph_0} \times Y$:

$$\big\{(x, t) : t \in Y \text{ and } x \in \varphi(t)\big\}$$

and let $\pi : I^{\aleph_0} \times Y \to Y$ be the projection. The restriction $f = \pi \restriction X : X \to Y$ is a perfect map (cf. [7, § 18] or [11, Exercise 1.11.26]) with hereditarily indecomposable fibers. By Theorem 1.1 there exists a hereditarily indecomposable n-dimensional compact space X^* that contains X and hence a copy of every hereditarily indecomposable continuum of dimension n.

The decomposition of X^* into its components yields a compact zero-dimensional space. The pseudo-arc P contains a copy of this decomposition space (as indeed does any uncountable compact metrizable space). Let $q : X^* \to P$ be a map such that $A = q[X]$ is that decomposition space and $q : X \to A$ is the quotient map.

By Theorem 15 of [9] there exist a hereditarily indecomposable continuum Z_n and an atomic mapping r from Z_n onto P such that $r \restriction r^{-1}(P \setminus A)$ is a homeomorphism and $r^{-1}(A)$ is homeomorphic to X^* (Z_n is a so-called pseudosuspension

of X^* over P by q). Since $\dim Z_n \leqslant n$ by the countable sum theorem and Z_n contains X^* topologically, the space Z_n has the required properties. $\qquad\square$

PROOF USING THEOREM 1.2. Use the second half of the previous proof but now take the pseudosuspension of the space $X(n, \aleph_0)$ over P by q, where $q : X(n, \aleph_0) \to P$ is a quotient map such that $A = q[X(n, \aleph_0)]$ is the decomposition space of $X(n, \aleph_0)$ into its components. $\qquad\square$

There is an obvious difference between Maćkowiak's result and our Theorem 1.2: Maćkowiak obtained universal *continua* whereas we obtain universal hereditarily indecomposable *compacta*.

Question. Does there exist, for a given cardinal τ and natural number n, a universal hereditarily indecomposable *continuum* of weight τ and covering dimension n?

Remark. If one uses Theorem 2.3 instead of Mardešić's Factorization Theorem, and standard topological reasoning (see [3, proofs of Theorems 5.4.3 and 3.4.2]) one gets the following results.

Proposition 4.2. *For every hereditarily indecomposable compact space X such that $\dim X = n$ and the weight of X is equal to τ, there exists an inverse system $\mathbf{S} = \{X_\sigma, \pi_\rho^\sigma, \Sigma\}$, where $|\Sigma| \leqslant \tau$, of metrizable hereditarily indecomposable compact spaces of dimension n whose limit is homeomorphic to X. If X is a continuum, then all X_σ are continua.*

Proposition 4.3. *Every normal n-dimensional space X of weight τ that has Property (KM) has a hereditarily indecomposable compactification $\tilde{X}$ of dimension n and of weight τ.*

Remark. The results of this paper remain valid if in the formulation of Property (KM) one replaces closed sets by zero-sets and open sets by cozero-sets. This implies that in Theorem 2.1 one can relax the assumption of normality to complete regularity.

REFERENCES

[1] J. M. Aarts, *Wallman-Shanin Compactification*, Encyclopedia of general topology, (Klaas Pieter Hart, Jun-iti Nagata, and Jerry E. Vaughan, eds.), Elsevier Science Publishers B.V., Amsterdam, 2004, pp. 218–220.

[2] Paul Bankston, *Some applications of the ultrapower theorem to the theory of compacta*, Applied Categorical Structures, **8** (2000), no. 1-2, 45–66. Papers in honour of Bernhard Banaschewski (Cape Town, 1996). MR 1785837 (2001f:54011)

[3] Ryszard Engelking, *Theory of dimensions finite and infinite*, Sigma Series in Pure Mathematics, vol. 10, Heldermann Verlag, Lemgo, 1995. MR 1363947 (97j:54033)

[4] Klaas Pieter Hart, Jan van Mill, and Roman Pol, *Remarks on hereditarily indecomposable continua*, Topology Proceedings, **25** (2000), 179–206 (2002). MR 1925683 (2003k:54028)

[5] Wilfrid Hodges, *A shorter model theory*, Cambridge University Press, Cambridge, 1997. MR 98i:03041

[6] Józef Krasinkiewicz and Piotr Minc, *Mappings onto indecomposable continua*, Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques, **25** (1977), 675–680. MR 0464184 (57 #4119)

[7] K. Kuratowski, *Topology. Vol. I*, Academic Press, New York, 1966. MR 0217751 (36 #840)

[8] ———, *Topology. Vol. II*, Academic Press, New York, 1969. MR 0259835 (41 #4467)

[9] T. Maćkowiak, *The condensation of singularities in arc-like continua*, Houston Journal of Mathematics, **11** (1985), 535–558. MR 837992 (87m:54099)

[10] ———, *A universal hereditarily indecomposable continuum*, Proceedings of the American Mathematical Society, **94** (1985), 167–172. MR 781076 (86j:54061)

[11] Jan van Mill, *The infinite-dimensional topology of function spaces*, North-Holland Mathematical Library, vol. 64, North-Holland Publishing Co. Amsterdam, 2001. MR 1851014 (2002h:57031)

[12] B. J. van der Steeg, *Models in Topology*, PhD thesis, TU Delft, 2003.

[13] Henry Wallman, *Lattices and topological spaces*, Annals of Mathematics, **39** (1938), 112–126. MR 1503392

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