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A small transitive family of continuous functions on the Cantor set

K.P. Hart *, B.J. van der Steeg

TU Delft, Faculty ITS, Postbus 5031, 2600 GA Delft, Netherlands

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Abstract

In this paper we show that, when we iteratively add Sacks reals to a model of ZFC we have for every two reals in the extension a continuous function defined in the ground model that maps one of the reals to the other.

1. Introduction

In [4] Dow gave a proof of the Rudin–Shelah theorem about the existence of $2^{\mathfrak{c}}$ points in $\beta\mathbb{N}$ that are Rudin–Keisler incomparable. The proof actually shows that whenever a family $\mathcal{F}$ of $\mathfrak{c}$ continuous self-maps of $\beta\mathbb{N}$ (or $\mathbb{N}^*$) are given there is a set S of $2^{\mathfrak{c}}$ many $\mathcal{F}$-independent points in $\beta\mathbb{N}$ (or $\mathbb{N}^*$). This suggests that we measure the complexity of a space X by the cardinal number $\mathfrak{tf}(X)$, defined as the minimum cardinality of a set $\mathcal{F}$ of continuous self maps such that for all $x, y \in X$ there is $f \in \mathcal{F}$ such that $f(x) = y$ or $f(y) = x$. Let us call such an $\mathcal{F}$ transitive. Thus Dow's proof shows $\mathfrak{tf}(\beta\mathbb{N})$, $\mathfrak{tf}(\mathbb{N}^*) \geqslant \mathfrak{c}^+$.

We investigate $\mathfrak{tf}(C)$, where C denotes the Cantor set. Van Mill observed that $\mathfrak{tf}(C) \geqslant \aleph_1$; a slight extension of his argument shows that MA(countable) implies $\mathfrak{tf}(C) = \mathfrak{c}$. Our main result states that in the Sacks model the continuous functions on the Cantor set that are coded in the ground model form a transitive set. Thus we get the consistency of $\mathfrak{tf}(C) = \aleph_1 < \aleph_2 = \mathfrak{c}$.

The gap between $\mathfrak{tf}(C)$ and $\mathfrak{c}$ cannot be arbitrarily wide, because Hajnal's free set lemma implies that for any space X one has $|X| \leqslant \mathfrak{tf}(X)^+$.

In [7] Miller showed that it is consistent with ZFC that for every set of reals of size continuum there is a continuous map from that set onto the closed unit interval. In fact he

* Corresponding author.
E-mail address: k.p.hart@its.tudelft.nl (K.P. Hart).

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showed that the iterated perfect set model of Baumgartner and Laver (see [2]) is such a model, and noted that the continuous map can even be coded in the ground model.

Here we will show that in the iterated perfect set model, for every two reals x and y there exists a continuous function with code in the ground model that maps x to y or y to x.

Definition 1. By a transitive set of functions $\mathcal{F}$ we mean a set of continuous functions such that for every two reals x and y there exists an element $f \in \mathcal{F}$ such that $f(x) = y$ or $f(y) = x$ holds.

Let us also define the cardinal number $\mathfrak{tf}$ by

$$\mathfrak{tf} = \min\{|\mathcal{F}|\colon \mathcal{F} \text{ is a transitive set of functions}\}, \quad \text{i.e.,} \quad \mathfrak{tf} = \mathfrak{tf}(C).$$

The paper is organized as follows: in Section 2 we prove some simple facts on $\mathfrak{tf}$, the minimal size of transitive sets of functions. We also state and prove the main theorem of this paper in Section 2, using theorems proved later on in Section 3. As a corollary to the main theorem we have the consistency of $\mathfrak{tf} < \mathfrak{c}$ with ZFC. Finally in Section 4 we will make a remark on the effect on $\mathfrak{tf}$ when we add κ many Sacks reals side-by-side to a model of ZFC+CH.

2. Notation and preliminaries

For the rest of this paper let V be a model of ZFC. We will use the same notations and definitions as Baumgartner and Laver in [2], so for any ordinal α we let $\mathbb{P}_\alpha$ denote the poset that iteratively adds α Sacks reals to the model V, using countable support. Let $\mathbb{P}_1 = \mathbb{P}$, where $\mathbb{P}$ denotes the 'normal' Sacks poset for the addition of one Sacks real.

Let G_α be $\mathbb{P}_\alpha$-generic over V, we define V_α by $V_\alpha = V[G_\alpha]$ for every ordinal α. Note that if $\beta < \alpha$ we have that $G_\alpha \restriction \beta$ is a $\mathbb{P}_\beta$-generic subset over V. If we denote the $(\alpha+1)$th added Sacks real by s_α then we can also write $V_\alpha = V[\langle s_\beta\colon \beta < \alpha\rangle]$.

Assuming $V \models \text{CH}$, the proof of the following facts can be found in [2]:

(1) Forcing with $\mathbb{P}_\alpha$ does not collapse cardinals.
(2) V_{ω_2} is a model of $\text{ZFC} + 2^{\aleph_0} = \aleph_2$.
(3) Let $\dot{\mathbb{P}}_\beta$ denote the result of defining $\mathbb{P}_\beta$ in V_α. Then for any $\alpha, \beta \geqslant 1$, $\Vdash_\alpha$ "$\mathbb{P}_{\alpha,\alpha+\beta}$ is isomorphic to $\dot{\mathbb{P}}_\beta$".

We will now prove some facts on the cardinal $\mathfrak{tf}$. The first is Van Mill's observation alluded to above.

Theorem 2. $\mathfrak{tf} \geqslant \aleph_1$.

Proof. Suppose $\mathcal{F}$ is a countable set of functions. Let A_f denote the set $\{x\colon \operatorname{int}(f^{-1}(x)) \neq \emptyset\}$ for every $f \in \mathcal{F}$. Every A_f is at most countable because 2^ω is separable. So choose an x in $2^\omega \setminus \bigcup_{f\in\mathcal{F}} A_f$, then we know that for every $f \in \mathcal{F}$ the set $f^{-1}(x)$ is nowhere dense in 2^ω. For such an x the set $\{f^{-1}(x)\colon f \in \mathcal{F}\}$ is countable. Because the set $\{f(x)\colon f \in \mathcal{F}\}$ is also countable the Baire category theorem tells us that the set $2^\omega \setminus \bigcup_{f\in\mathcal{F}}(\{f(x)\}\cup f^{-1}(x))$ is nonempty, thus showing that $\mathcal{F}$ is not transitive. □

Theorem 3. $\mathfrak{tf} \leqslant \mathfrak{c} \leqslant \mathfrak{tf}^+$.

Remark 4. The proof of Theorem 2 shows that $\mathfrak{tf}$ is at least the minimum number of nowhere dense sets needed to cover C. Then Theorem 3 and MA(countable) imply $\mathfrak{tf} = \mathfrak{c}$.

The second inequality is a consequence of the following lemma. The proof of this lemma can be found in [8].

For this we need some more notation. Let S be an arbitrary set. By a *set mapping on* S we mean a function f mapping S into the power set of S. The set map is said to be of *order* λ if λ is the least cardinal such that $|f(x)| < \lambda$ for each x in S. A subset S' of S is said to be *free for* f if for every $x \in S'$ we have $f(x) \cap S' \subset \{x\}$.

Lemma 5 (Free set lemma). *Let S be a set with $|S| = \kappa$ and f a set map on S of order λ where $\lambda < \kappa$. Then there is a free set of size κ for f.*

Proof of Theorem 3. The proof of the first inequality is easy. We simply have to observe that the set of all constant functions on the reals is a transitive set of functions.

Now for the second inequality. Striving for a contradiction, suppose that $\mathfrak{c} \geqslant \mathfrak{tf}^{++}$. Let $\mathcal{F}$ be a transitive set of functions such that $|\mathcal{F}| = \mathfrak{tf}$. We define a set map F on the reals by $F(x) = \{f(x)\colon f \in \mathcal{F}\}$ for every $x \in 2^\omega$. Because $|F(x)| \leqslant \mathfrak{tf}$, this set map F is of order $\mathfrak{tf}^+$, which is less than $\mathfrak{c}$. According to the free set lemma there exists a set $X \subset 2^\omega$ such that $|X| = \mathfrak{c}$ and for every $x \in X$ we have $F(x) \cap X \subset \{x\}$. This is a contradiction, because every two reals in X provide a counter example of $\mathcal{F}$ being a transitive set. □

Closed subsets of the Cantor set can be coded by sub-trees of ${}^{<\omega}2$, as follows: if A is closed then let $T_A = \{x \restriction n\colon x \in A,\ n \in \omega\}$; one can recover A from T_A by observing that $A = \{x \in {}^\omega 2\colon \forall n \in \omega,\ x \restriction n \in T_A\}$.

When we say that a closed set A is *coded in the ground model* we mean that T_A belongs to the ground model.

We shall always construct a continuous function f between closed sets A and B by specifying an order-preserving map ϕ from T'_A to T_B, where T'_A denotes the set of splitting nodes of T_A. Once ϕ is found one defines f by

$$f(x) = \text{“the path through } T_B \text{ determined by the restriction of } \phi \text{ to } \{x \restriction n\colon n \in \omega\}\text{”}.$$

We say that f is coded in the ground model if ϕ belongs to V. In what follows we shall denote the map ϕ by f as well.

Let us define the set $\mathcal{G}$ (in any V_α) by

$$\mathcal{G} = \{f\colon f \text{ is a continuous function with code in } V\}.$$

Now we can explicitly state the main theorem of this paper. Section 3 is completely devoted to the proof of this theorem by parts, so we will prove the theorem here and refer to the needed theorems proved in that section.

Theorem 6 (Main Theorem). *The set $\mathcal{G}$ is transitive in V_α for every ordinal α.*

Proof. We will show by transfinite induction that $\mathcal{G}$ is a transitive set in V_α for all α. For $\alpha = 0$ this is obvious. Suppose the theorem is true for all $\beta < \alpha$. Let x and y be reals in V_α.

If α is a successor ordinal, $\alpha = \beta + 1$, then we use Theorem 11 in the case that at least one of x and y is not in V_β to show that there exist a continuous function f defined in V (so $f \in \mathcal{G}$) such that in V_α we have $f(x) = y$ or $f(y) = x$.

Since we are forcing with countable support and because reals are countable objects, there are no new reals added by $\mathbb{P}_\alpha$ for $\mathrm{cf}(\alpha) > \aleph_0$. So if α is a limit ordinal we only have to consider the case where $\mathrm{cf}(\alpha) = \aleph_0$ and at least one of x, y is not in $\bigcup_{\beta<\alpha} V_\beta$. Then we use Theorem 17 to show the existence of an continuous function f defined in V such that in V_α $f(x) = y$ or $f(y) = x$ holds. □

As is well known, if $V \models \mathrm{CH}$ then $V_{\omega_2} \models \mathfrak{c} = \aleph_2$. This enables us to show that $\mathfrak{tf} < \mathfrak{c}$ is consistent.

Corollary 7. *If $V \models \mathrm{CH}$ then $V_{\omega_2} \models \mathfrak{tf} < \mathfrak{c}$.*

In this paper we shall repeatedly use the fact that any homeomorphism h between two closed nowhere dense subsets of the Cantor set can be extended to a homeomorphism of the Cantor set onto itself (see [6]). Furthermore it is straightforward to extend a continuous function between to closed nowhere dense (disjoint) subsets of the Cantor set to a continuous self map of the Cantor set.

Because we can make sure that the subsets of the Cantor set that define the added reals x and y are nowhere dense and closed, when we show that there exists a homeomorphism (or a continuous function) f mapping of one of these sets onto the other, in such a way that in the extension x is mapped to y or vice versa, we actually have shown that there exists a self map of the Cantor that is a homeomorphism (continuous function) mapping, in the extension, x to y or y to x.

3. The continuous functions with code in the ground model V form a transitive set in V_α

In this section we prove that for every α and any new real x in the Baumgartner and Laver model V_α (i.e., $x \in V_\alpha \setminus \bigcup_{\beta<\alpha} V_\beta$) and y any real in V_α there exists a function f defined in the ground model V such that in V_α the equation $f(x) = y$ holds.

We make the following definition. For any $\sigma \in {}^{<\omega}2$ we let $l(\sigma) \in \omega$ denote the *length of* σ. So for every $\sigma \in {}^{<\omega}2$ we have $\sigma \in {}^{l(\sigma)}2$. To show how we construct our continuous maps we reprove the familiar fact that Sacks reals are minimal, see [5].

Lemma 8. *Suppose x is a real in $V[G] \setminus V$, where G is a $\mathbb{P}$-generic filter over V, and that $p \in \mathbb{P}$ is such that $p \Vdash$ "$\dot{x} \notin V$". Then there exists a $q \geqslant p$ and a homeomorphism f defined in V such that $q \Vdash$ "$f(\dot{s}) = \dot{x}$". Here $\dot{s}$ denotes the name of the added Sacks real.*

Proof. We will construct a fusion sequence $\{(p_i, n_i)\colon i \in \omega\}$ such that each p_{i+1} will know all the first i splitting nodes of every branch of the perfect tree p_i and $(p_{i+1}, n_{i+1}) > (p_i, n_i)$ for every i.

Because p forces that $\dot{x}$ is a new real, there exists an element $u_\emptyset \in {}^{<\omega}2$ with maximal length $m_\emptyset$, such that $p \Vdash$ "$\dot{x} \restriction m_\emptyset = u_\emptyset$" and p does not decide $\dot{x}(m_\emptyset)$. There exist $p_{\langle 0\rangle}, p_{\langle 1\rangle} \geqslant p_0$ such that $p_{\langle k\rangle} \Vdash$ "$\dot{x}(m_\emptyset) = k$" for $k \in \{0, 1\}$. Without loss of generality the stems of $p_{\langle 0\rangle}$ and $p_{\langle 1\rangle}$ are incompatible. Let $n_0 = \min\{n \in \omega\colon p_{\langle 0\rangle} \restriction n \neq p_{\langle 1\rangle} \restriction n\}$ and let p_0 denote the element $p_{\langle 0\rangle} \cup p_{\langle 1\rangle}$.

Now assume we have $p_i = \bigcup\{p_\sigma\colon \sigma \in {}^{i+1}2\}$. Consider $\tau \in {}^{i+1}2$, we have an element $u_\tau \in {}^{<\omega}2$ of maximal length m_τ such that $p_\tau \Vdash$ "$\dot{x} \restriction m_\tau = u_\tau$". There exist $p_{\tau\frown 0}$, $p_{\tau\frown 1} \geqslant p_\tau$ such that $p_{\tau\frown k} \Vdash$ "$\dot{x}(m_\tau) = k$" for $k \in \{0, 1\}$. Again without loss of generality the stems of $p_{\tau\frown 0}$ and $p_{\tau\frown 1}$ are incompatible. Let n_τ denote the integer $\min\{n \in \omega\colon p_{\tau\frown 0} \restriction n \neq p_{\tau\frown 0} \restriction n\}$ and $n_{i+1} = \max\{n_\sigma\colon \sigma \in {}^{i+1}2\}$. We let p_{i+1} denote the element $\bigcup\{p_\sigma\colon \sigma \in {}^{i+2}2\}$. Now the induction step is completed, because p_{i+1} knows all the first $i+1$ splitting nodes of every branch in p_i and $(p_{i+1}, n_{i+1}) > (p_i, n_i)$ for every $i \in \omega$.

We define the function f by

$$f^{-1}\bigl([u_\sigma]\bigr) \supset \bigl[\mathrm{stem}(p_\sigma)\bigr] \quad \text{for } \sigma \in {}^{<\omega}2.$$

As $\mathrm{stem}(p_\sigma)$ is a finite approximation of the added Sacks real $\dot{s}$, we have by the construction of our p_σ for $\sigma \in {}^{<\omega}2$ and the function f that $p_\sigma \Vdash$ "$f(\dot{s}) \in [u_\sigma]$" for every $\sigma \in {}^{<\omega}2$. And so the fusion q of the sequence $\{(p_i, n_i)\colon i \in \omega\}$ forces that in the extension $V[G]$ the equality $f(s) = x$ holds. This f, being a continuous bijection between two Cantor sets, is (of course) a homeomorphism. □

Remark 9. In the lemma we have also defined a map ϕ from the finite sub-trees of the fusion q to the finite sub-trees of $T = \bigcup_{\sigma \in {}^{<\omega}2} u_\sigma$ which induces our homeomorphism. We have $\phi(q) = T$ and

$$\phi\bigl([q \restriction \sigma]\bigr) = \bigcup\bigl\{u_\tau\colon \sigma \subset \tau \text{ and } \tau \in {}^{<\omega}2\bigr\}.$$

We note that $[T]$ is the set of all the possible interpretations of $\dot{x}$ in $V[G]$ and that T depends on ϕ and q only. In Theorem 11 we will use this interpretation of the previous lemma.

As a warming up exercise we prove the following.

Theorem 10. *The set $\mathcal{G}$ is transitive in V_1.*

Proof. Suppose x and y are two reals of $V_1 (= V[s_0])$. We consider two cases.

Case 1. x is a real in V. The constant function $c_x = \{\langle y, x\rangle\colon\ y \text{ a real in } V_1\}$ is a continuous function defined in V, thus a member of $\mathcal{G}$, and in V_1 it maps y onto x.

Case 2. Both x and y are reals not in V. Let $p \in \mathbb{P}$ be a witness of this, so $p \Vdash$ "$\dot{x}, \dot{y} \notin V$". According to Lemma 8 there exists a $q \geqslant p$ and a homeomorphism f defined in V such that $q \Vdash$ "$f(\dot{s}_0) = \dot{x}$", where $\dot{s}_0$ denotes the added Sacks real. If we apply the lemma again we get an $r \geqslant q$ and a homeomorphism g defined in V such that $q \Vdash$ "$g(\dot{s}_0) = \dot{y}$". But now we have that $r \Vdash$ "$(g \circ f^{-1})(\dot{x}) = \dot{y}$" and we see that $g \circ f^{-1}$ is the element of $\mathcal{G}$ we are looking for. □

Theorem 11. *For α an ordinal and x and y reals in $V_{\alpha+1}$ such that $x \notin V_\alpha$ there exists an $f \in \mathcal{G}$ such that in $V_{\alpha+1}$ $f(x) = y$ holds.*

Moreover if also $y \notin V_\alpha$ then f can be chosen to be a homeomorphism.

Proof. This is an immediate consequence of Lemmas 14 and 15. □

We make the following definitions. For $p \in \mathbb{P}$ and $s \in {}^{<\omega}2$ we let p_s denote the sub-tree $\{t \in p\colon\ s \subseteq t \text{ or } t \subseteq s\}$ of p. Of course p_s is a perfect tree if and only if $s \in {}^{<\omega}2 \cap p$. To generalize this to $\mathbb{P}_\alpha$, suppose p is an element of $\mathbb{P}_\alpha$, F is a finite subset of $\mathrm{dom}(p)$ and $n \in \omega$, we say that a function $\tau : F \to {}^n2$ *is consistent with* p if the following holds for every $\beta \in F$:

$$(p \restriction \tau) \restriction \beta \Vdash_\beta \text{“}\tau(\beta) \in p(\beta)\text{”}.$$

So we have for every $\beta \in F$ that $(p \restriction \tau) \restriction \beta \Vdash_\beta$ "$(p(\beta))_{\tau(\beta)}$ is a perfect tree".

Furthermore let us suppose that F and H are two sets such that $F \subset H$, and n and m are two integers such that $m < n$, if τ is a function mapping F into ${}^m2$ then we say that a function $\sigma : H \to {}^n2$ *extends* the function τ if for every $i \in F$ we have $\sigma(i) \restriction m = \tau(i)$.

For later use we will prove the following:

Lemma 12. *Let $p \in \mathbb{P}_\alpha$, $F \in [\mathrm{dom}(p)]^{<\omega}$ and $n \in \omega$. Suppose $\tau : F \to {}^n2$ is consistent with p then for every $r \geqslant p \restriction \tau$ there exists a $q \geqslant p$ such that $q \restriction \tau = r$ and $q \restriction \beta \Vdash_\beta$ "$(p(\beta))_s = (q(\beta))_s$ for every $s \in {}^n2$ such that $s \neq \tau(\beta)$" for every $\beta \in F$.*

Proof. Define the element $q \in \mathbb{P}_\alpha$ as follows for $\beta < \alpha$:

$$q \restriction \beta \Vdash_\beta \text{“}q(\beta) = \begin{cases} r(\beta), & \beta \notin F, \\ r(\beta) \cup \big\{(p(\beta))_s\colon\ s \in {}^n2 \cap p(\beta) & \\ \qquad\qquad \text{such that } s \neq \tau(\beta)\big\}, & \beta \in F\text{”}. \end{cases}$$

In this way we strengthen the tree $p(\beta)$ above $\tau(\beta)$ keeping the rest of the perfect tree intact (according to F anyway). □

We need the following lemma to make sure that the maps we will construct in Lemmas 14 and 15 are well-defined and continuous.

Lemma 13. *Let $p \in \mathbb{P}_{\alpha+1}$. Suppose $F, H \in [\mathrm{dom}(p)]^{<\omega}$ are such that $F \subset H$ and $m, n \in \omega$ are such that $m < n$. If $\tau : F \to {}^m 2$ is consistent with p, N is an integer and T is a finite tree such that*

$$(p \restriction \tau) \restriction \alpha \Vdash \text{“}p(\alpha) \cap {}^{\leqslant N} 2 = T\text{”},$$

then there exist a $(q, j) >_H (p \restriction \tau, n)$ and an $M > N$ such that for every $\sigma : H \to {}^n 2$ extending τ, if σ is consistent with q, then there exists T_σ such that $q \restriction \sigma \Vdash$ “$q(\alpha) \cap {}^{\leqslant M} 2 = T_\sigma$”. Also $|(T_\sigma)_t \cap {}^M 2| \geqslant 2$ for every $t \in T$ and $[T_\sigma] \cap [T_\varsigma] = \emptyset$ whenever σ and ς are distinct and consistent with q.

Proof. Let Σ_τ denote the set of all $\sigma : H \to {}^n 2$ extending τ. Because $p(\alpha)$ is a perfect tree there exists a $\mathbb{P}_\alpha$-name $\dot{M}$ such that for every $t \in T$ we have

$$(p \restriction \tau) \restriction \alpha \Vdash \text{“}\bigl|\bigl(p(\alpha)\bigr)_t \cap {}^{\dot{M}} 2\bigr| \geqslant 2|\Sigma_\tau|\text{”}.$$

According to Lemma 2.3 of [2] there exists a $(q^\dagger, j^\dagger) >_H ((p \restriction \tau) \restriction \alpha, n)$ such that if $\sigma \in \Sigma_\tau$ is consistent with $q^\dagger$ we have an M_σ such that $q^\dagger \restriction \sigma \Vdash$ “$\dot{M} = M_\sigma$”. Put $M = \max\{M_\sigma \colon \sigma \in \Sigma_\tau$ consistent with $q^\dagger\}$. We have $q^\dagger \Vdash$ “$|(p(\alpha))_t \cap {}^M 2| \geqslant 2|\Sigma_\tau|$” for every $t \in T$.

Enumerate $\{\sigma \in \Sigma_\tau \colon \sigma$ consistent with $q^\dagger\}$ as $\{\sigma_k \colon k < K\}$. Let $r \geqslant q^\dagger \restriction \sigma_0$ be such that $r \Vdash$ “$p(\alpha) \cap {}^{\leqslant M} 2 = S_{\sigma_0}$”, where S_{σ_0} is such that $|(S_{\sigma_0})_t \cap {}^M 2| \geqslant 2|\Sigma_\tau|$ for every $t \in T$. Use Lemma 12 to find a $q_0 \geqslant q^\dagger$ such that $q_0 \restriction \sigma_0 = r$.

We continue this procedure with all the $\sigma_k \in \Sigma_\tau$. So if σ_k is consistent with q_{k-1} we find an $r \geqslant q_{k-1} \restriction \sigma_k$ such that $r \Vdash$ “$p(\alpha) \cap {}^{\leqslant M} 2 = S_{\sigma_k}$”, and also that $|(S_{\sigma_k})_t \cap {}^M 2| \geqslant 2|\Sigma_\tau|$ for every $t \in T$. And we use Lemma 12 to define $q_k \geqslant q_{k-1}$ such that $q_k \restriction \sigma_k = r$. If σ_k is not consistent with q_{k-1} we choose $q_k = q_{k-1}$.

We now have for every $\sigma \in \Sigma_\tau$ consistent with q_{K-1} a finite tree $S_\sigma \subset {}^{\leqslant M} 2$ extending the tree T such that every branch in T has (at least) $2|\Sigma_\tau|$ different extensions in $S_\sigma \cap {}^M 2$ and $q_{K-1} \restriction \sigma \Vdash$ “$p(\alpha) \cap {}^{\leqslant M} 2 = S_\sigma$”.

As q_{K-1} forces that, for each $y \in T$ the size of the set $p(\alpha)_t \cap {}^M 2$ is at least $2|\Sigma_\tau|$ we can find for $\sigma \in \Sigma_\tau$ consistent with q_{K-1} a sub-tree T_σ of S_σ such that $|(T_\sigma)_t \cap {}^M 2| \geqslant 2$ and whenever σ and ς are distinct and consistent with q_{K-1} we have $[T_\sigma] \cap [T_\varsigma] = \emptyset$.

Define $q \in \mathbb{P}_{\alpha+1}$ such that $q \restriction \alpha = q_{K-1}$ and choose $q(\alpha)$ such that for every consistent $\sigma \in \Sigma_\tau$ we have $q \restriction \sigma \Vdash$ “$q(\alpha) = p(\alpha) \cap [T_\sigma]$”. If we let j be equal to $\max\{j^\dagger, M\}$ the proof is complete. □

Lemma 14. *Given an ordinal α, a $p \in \mathbb{P}_{\alpha+1}$ and $\mathbb{P}_{\alpha+1}$-names $\dot{x}$ and $\dot{y}$ such that $p \Vdash$ “$\dot{x} \notin V_\alpha$ and $\dot{y} \in V_\alpha$” then there exists a continuous function f defined in V and a $q \geqslant p$ such that $q \Vdash$ “$f(\dot{x}) = \dot{y}$”.*

Proof. By Remark 9 we know that there is an $r \geqslant p \restriction \alpha$ and there exist $\mathbb{P}_{\alpha+1}$ names $\dot{\phi}$ for a map on the finite sub-trees of $p(\alpha)$ and $\dot{T}$ for a perfect tree such that $r \Vdash$ "$\dot{\phi}(p(\alpha)) = \dot{T}$". Without loss of generality we assume that $p \restriction \alpha = r$.

Let us construct a fusion sequence $\{\langle p_i, n_i, F_i\rangle\colon i \in \omega\}$. Let $p_0 = p_1 = p$, $n_0 = n_1 = 0$, $F_0 = \emptyset$ and choose $F_1 \in [\mathrm{dom}(p)]^{<\omega}$ in such a way that we are building a fusion sequence.

Suppose we have constructed the sequence up to i, let us construct the next element of the fusion sequence. We let $\{\tau_k\colon k < K\}$ denote all $\tau : F_{i-1} \to {}^{n_{i-1}}2$ consistent with p_i. If we choose in Lemma 13 $\tau = \tau_0$, $F = F_{i-1}$ and $m = n_{i-1}$ we get a $(q_0, m_0) >_{F_i} (p_i \restriction \tau_0, n_i)$ such that for every $\sigma : F_i \to {}^{\leqslant n_i}2$ extending τ_0, consistent with q_0, we have a finite sub tree $T_\sigma \subset {}^{\leqslant M(\tau_0)}2$ ($M(\tau_0) \in \omega$ follows from Lemma 13) of $p_i(\alpha) = p(\alpha)$ such that

(1) T_σ is an extension of T_{τ_0},
(2) for every branch t in T_{τ_0} there exist at least two different branches of length $M(\tau_0)$ in T_σ extending t,
(3) if σ and ς are two distinct members of Σ_{τ_0} consistent with q_0 we have $[T_\sigma] \cap [T_\varsigma] = \emptyset$.

We choose $r_0 \in \mathbb{P}_{\alpha+1}$ with Lemma 12 such that $r_0 \geqslant q_0$ and $r_0 \restriction \tau_0 = q_0$.

We iteratively consider all the $\tau : F_{i-1} \to {}^{n_{i-1}}2$. In the general case if τ_k is consistent with r_{k-1} then Lemma 13 gives us a q_k and an $m_k \in \omega$ such that $(q_k, m_k) >_{F_i} (r_{k-1} \restriction \tau_k, n_i)$. We choose r_k in the same way as above, using Lemma 12 such that $r_k \geqslant q_k$ and $r_k \restriction \tau_k = q_k$. If τ_k is inconsistent with r_{k-1} then we choose $r_k = r_{k-1}$ and $m_k = m_{k-1}$. After considering all the τ_k's we define $p_{i+1} = r_{K-1}$ and $n_{i+1} = \max\{m_k\colon k < K\}$. This ends the construction of the next element of the fusion sequence.

For every $i < \omega$ if $\sigma : F_i \to {}^{n_i}2$ is consistent with p_{i+1} and extends $\tau : F_{i-1} \to {}^{n_{i-1}}2$ then

$$p_{i+1} \restriction \sigma \Vdash \text{“}p(\alpha) \cap {}^{\leqslant M(\tau)}2 = T_\sigma\text{”}.$$

Considering our function $\dot{\phi}$, let us denote the finite tree $\dot{\phi}(T_\sigma)$ by S_σ. We have

$$p_{i+1} \restriction \sigma \Vdash \text{“}\dot{\phi}(T_\sigma) = S_\sigma\text{”}.$$

When we are building the fusion sequence we can of course make sure that the fusion determines $\dot{y}$ as well. Suppose we have that $p_i \restriction \tau_k \Vdash$ "$t_{\tau_k} \subset \dot{y}$", t_{τ_k} of length $i+1$. With Lemma 13 we can choose q_k strong enough such that for every $\sigma \in \Sigma_{\tau_k}$ consistent with q_k we have a t_σ of length $i+2$ such that $q_k \restriction \sigma \Vdash$ "$t_\sigma \subset \dot{y}$". So assume we have made sure this is the case and let us define the function f in V by $f(b) = t_\sigma$ for every maximal branch $b \in S_\sigma$ for every $\sigma : F_i \to {}^{n_i}2$ consistent with p_i for some $i \in \omega$. The function f is well-defined by Lemma 13 and we have for every $i \in \omega$ and $\sigma : F_i \to {}^{n_i}2$ consistent with p_i that $p_i \restriction \sigma \Vdash$ "$f([S_\sigma]) \subset [t_\sigma]$" and thus $q \Vdash$ "$f(\dot{x}) = \dot{y}$". □

Lemma 15. *Given an ordinal α, a $p \in \mathbb{P}_{\alpha+1}$ and $\mathbb{P}_{\alpha+1}$-names $\dot{x}$ and $\dot{y}$ such that $p \Vdash$ "$\dot{x}, \dot{y} \notin V_\alpha$" then there exists a homeomorphism f, with code in V, and a $q \geqslant p$ such that $q \Vdash$ "$f(\dot{x}) = \dot{y}$".*

Proof. By applying Remark 9 twice we have an $r \geqslant p$ in $\mathbb{P}_{\alpha+1}$ and $\mathbb{P}_{\alpha+1}$ names $\dot\phi_x$, $\dot\phi_y$ and $\dot T_x$, $\dot T_y$ for maps and perfect trees respectively such that $r \restriction \alpha \Vdash$ "$\dot\phi_x(p(\alpha)) = \dot T_x$ and $\dot\phi_y(p(\alpha)) = \dot T_y$". Without loss of generality we can assume that $p \restriction \alpha = r$.

During the construction of possible finite sub-trees $(T_x)_\sigma$ for $\dot x$, when constructing the fusion sequence in the proof of Lemma 14 we could of course at the same time also have constructed a similar sequence of finite sub-trees $(T_y)_\sigma$ for $\dot y$.

Without loss of generality we could also have made sure that in the proof of Lemma 14 item 2 is replaced by

($2^\dagger$) for every maximal branch t in T_{τ_0} there are exactly two different branches of length $M(\tau_0)$ in T_σ extending t.

Following the proof of Lemma 14 we have for every $\sigma : F_i \to {}^{n_i}2$ consistent with p_{i+1} finite sub-trees S^x_σ and S^y_σ such that

$$p_{i+1} \restriction \sigma \Vdash \text{“}\dot\phi_x\big((T_x)_\sigma\big) = S^x_\sigma \quad \text{and} \quad \dot\phi_y\big((T_y)_\sigma\big) = S^y_\sigma.$$

We are ready to define the homeomorphism f in V that maps x to y in the extension. Suppose $\tau : F_i \to {}^{n_i}2$ and $\sigma : F_{i+1} \to {}^{n_{i+1}}2$ such that σ extends τ. Every maximal branch in $(T_x)_\tau$ corresponds to exactly one maximal branch in $(T_y)_\tau$. Let f map the splitting point in $(T_x)_\sigma$ above any maximal branch in $(T_x)_\tau$ to the splitting point in $(T_y)_\sigma$ above the corresponding maximal branch in $(T_y)_\tau$. The function f thus defined will be a continuous and one-to-one mapping between two Cantor sets, so a homeomorphism. Furthermore the fusion q forces that f maps x to y in the extension. □

Lemma 16. *Suppose that α is a limit ordinal of cofinality $\aleph_0$. Let x be a real in V_α such that $x \notin \bigcup_{\beta<\alpha} V_\beta$, and let $p \in \mathbb{P}_\alpha$ be a witness of this. Also let $F, H \in [\mathrm{dom}(p)]^{<\omega}$ such that $F \subset H$ and let n and m be two integers such that $m < n$. If $\tau : F \to {}^m2$ is consistent with p, and $u_\tau \in {}^{<\omega}2$ is such that*

$$p \restriction \tau \Vdash \text{“}u_\tau \subset \dot x\text{”},$$

then there exists a $(q, j) >_H (p \restriction \tau, n)$ such that for every $\sigma : H \to {}^n2$ consistent with q, we have a $u_\sigma \in {}^{<\omega}2$ such that $q \restriction \sigma \Vdash$ "$u_\sigma \subset \dot x$"; in addition we have $l(u_\sigma) = l(u_\varsigma)$ and $u_\sigma \neq u_\varsigma$ whenever σ and ς are distinct and consistent with q.

Before we prove the lemma we need some more notation. We let $\Vdash^*$ denote forcing in V_δ over $\mathbb{P}_{\delta\alpha}$. Here we use again the same notation as in [2] where for $\delta < \alpha$ $P_{\delta\alpha} = \{p \in \mathbb{P}_\alpha\colon \mathrm{dom}(p) \subset \{\xi\colon \delta \leqslant \xi < \alpha\}\}$, and if $p \in \mathbb{P}_\alpha$ then $p^\delta = p \setminus (p \restriction \delta) \in \mathbb{P}_{\delta\alpha}$. The mapping which carries p into $(p \restriction \delta, p^\delta)$ is an isomorphism of $\mathbb{P}_\alpha$ to a dense subset of $\mathbb{P}_\delta \times \mathbb{P}_{\delta\alpha}$ (see [2]).

Proof of Lemma 16. Choose a δ such that $\max(H) < \delta < \alpha$. Let $\tau : F \to {}^m2$ be consistent with p and let Σ_τ denote all the τ extending functions $\sigma : H \to {}^n2$.

Because p forces that $x \notin V_\delta$, there is an antichain below p^δ of size $|\Sigma_\tau|$ such that all these elements force different interpretations of $\dot x$ in the extension. In other words there exist a sequence $\{\dot{f}_\sigma\colon \sigma \in \Sigma_\tau\}$ of $\mathbb{P}_\delta$ names for elements of $\mathbb{P}_{\delta\alpha}$ and a sequence $\{\dot{u}_\sigma\colon \sigma \in \Sigma_\tau\}$ of $\mathbb{P}_\delta$ names for elements of ${}^{<\omega}2$ such that for all $\sigma \in \Sigma_\tau$ we have

$$(p \restriction \tau) \restriction \delta \Vdash \text{“}\dot{f}_\sigma \geqslant p^\delta \text{ and } \dot{f}_\sigma \Vdash^* \text{“}\dot{u}_\sigma \subset \dot{x}\text{””}, \tag{1}$$

and if σ and ς are distinct then

$$(p \restriction \tau) \restriction \delta \Vdash \text{“}l(\dot{u}_\sigma) = l(\dot{u}_\varsigma) \text{ and } \dot{u}_\sigma \neq \dot{u}_\varsigma\text{”}. \tag{2}$$

Repeatedly using Lemma 2.3 of [2] we see that there exist a $(q^\dagger, j) >_H ((p \restriction \tau) \restriction \delta, n)$ and sequences $\{f_\sigma\colon \sigma \in \Sigma_\tau\}$, $\{u_\sigma\colon \sigma \in \Sigma_\tau\} \subset {}^i2$ for some integer i such that for every $\sigma \in \Sigma_\tau$ we have

$$q^\dagger \Vdash_\delta \text{“}\dot{f}_\sigma = f_\sigma \text{ and } \dot{u}_\sigma = u_\sigma\text{”}. \tag{3}$$

Now let q denote the element of $\mathbb{P}_\alpha$ such that $q \restriction \delta = q^\dagger$, and $(q \restriction \sigma) \restriction \delta \Vdash$ “$q^\delta = f_\sigma$” for every $\sigma \in \Sigma_\tau$ consistent with $q^\dagger$. This completes the proof. □

Theorem 17. *For α a limit ordinal of cofinality $\aleph_0$ and x and y reals in V_α such that $x \notin \bigcup_{\beta<\alpha} V_\beta$, there exist a continuous function f defined in V such that in V_α the equation $f(x) = y$ holds.*

If also $y \notin \bigcup_{\beta<\alpha} V_\beta$ then f can be chosen to be a homeomorphism.

Proof. For the first part of the theorem suppose that we have $p \in \mathbb{P}_\alpha$ such that $p \Vdash$ “$\dot{x} \notin \bigcup_{\beta<\alpha} V_\beta$ and $\dot{y} \in \bigcup_{\beta<\alpha} V_\beta$”. We will construct a fusion sequence below p and define a continuous function f in V such that the fusion of the sequence forces that $f(x) = y$ holds in V_α.

Let $p_0 = p_1 = p$, $n_0 = n_1 = 0$, $F_0 = \emptyset$, and choose $F_1 \in [\mathrm{dom}(p)]^{<\omega}$ in such a way that we are building a fusion sequence. Suppose we have constructed the sequence up to i, we will construct the next element of the fusion sequence. Let $\{\tau_k\colon k < K\}$ denote an enumeration of all maps from F_{i-1} into ${}^{n_{i-1}}2$ consistent with p_i.

According to Lemma 16 there exists a $(q_0, j_0) >_{F_i} (p_i \restriction \tau_0, n_i)$ such that for every $\sigma : F_i \to {}^{n_i}2$ consistent with q_0 we have distinct u_σ's in ${}^{m(\tau_0)}2$ (where $m(\tau_0)$ follows from Lemma 16), such that $q_0 \restriction \sigma \Vdash$ “$u_\sigma \subset \dot{x}$”. Now use Lemma 12 to construct $r_0 \in \mathbb{P}_\alpha$ such that $r_0 \geqslant q_0$ and $r_0 \restriction \tau_0 = q_0$.

We now iteratively consider all the τ_k. In the general case if τ_k is not consistent with r_{k-1} then we make sure that $r_k = r_{k-1}$ and $j_k = j_{k-1}$. If τ_k is consistent with r_{k-1} we find by Lemma 16 a $(q_k, j_k) >_{F_i} (r_{k-1} \restriction \tau_k, n_i)$ such that for every $\sigma : F_i \to {}^{n_i}2$ consistent with q_k we have distinct u_σ's in ${}^{m(\tau_k)}2$ such that $q_k \restriction \sigma \Vdash$ “$u_\sigma \subset \dot{x}$”. Now use Lemma 12 to construct $r_k \in \mathbb{P}_\alpha$ such that $r_k \geqslant r_{k-1}$ and $r_k \restriction \tau_k = q_k$. After considering all τ_k we define $p_{i+1} = r_{K-1}$ and $n_{i+1} = \max\{j_k\colon k < K\}$.

If we take a closer look at Lemma 16 we can also let the fusion sequence that we just constructed determine $\dot{y}$. Because if we have $p \restriction \tau \Vdash$ “$t_\tau \subset \dot{y}$”, following the proof of Lemma 16 we can make sure that (by some strengthening of $q^\dagger$ or the f_σ's, if necessary) there exist t_σ's in ${}^{<\omega}2$, not necessarily distinct, extending t_τ such that for $\sigma : H \to {}^n2$

consistent with q we also have $q \restriction \sigma \Vdash$ "$t_\sigma \subset \dot{y}$". So assume we have done this. We have for every $\sigma : F_i \to {}^{n_i}2$ consistent with p_{i+1}

$$p_{i+1} \restriction \sigma \Vdash \text{"}u_\sigma \subset \dot{x} \text{ and } t_\sigma \subset \dot{y}\text{"}. \tag{4}$$

Now we are ready to define our function f which will map x in V_α continuously onto y. Let $f([u_\sigma]) \subset [t_\sigma]$ for all $\sigma : F_i \to {}^{n_i}2$ and all $i \in \omega$. Then $p_i \restriction \sigma \Vdash$ "$f(\dot{x}) \in [t_\sigma]$" for $\sigma : F_i \to {}^{n_i}2$ consistent with p_i and $i \in \omega$. It follows that the fusion q forces that in V_α we have $f(x) = y$. Moreover f is a continuous function, this follows from Lemma 16.

For the second part of the theorem suppose that $p \Vdash$ "$\dot{x}, \dot{y} \notin \bigcup_{\beta<\alpha} V_\beta$". Just as in Lemma 16 we can choose not only the u_σ's in Eq. (4) distinct but also the t_σ's for $\sigma \in \Sigma_\tau$ and $\tau : F_i \to {}^{n_i}2$ for some $i \in \omega$. With this, the constructed continuous function f is actually a homeomorphism. □

As there are no reals added at limit stages of cofinality larger than $\aleph_0$ we have as a corollary to Theorems 11 and 17.

Corollary 18. *For every α and every $\dot{x}$ and $\dot{y}$ $\mathbb{P}_\alpha$-names for reals in $V_\alpha \setminus \bigcup_{\beta<\alpha} V_\beta$ there exists a homeomorphism f defined in V such that in V_α we have $f(x) = y$.*

Remark 19. It is not the case that the $\mathfrak{tf}$ number is the same for all compact metric spaces, e.g., every Cook continuum X has $\mathfrak{tf}(X) = \mathfrak{c}$ (it only has the identity and constant mappings as self-maps, see [3]). On the other hand, in the Sacks model one has $\mathfrak{tf}(C) = \mathfrak{tf}(\mathbb{R}) = \mathfrak{tf}([0,1]) = \aleph_1$. To see this, observe that our proof produces, given x and y, two copies of the Cantor set A and B containing x and y respectively and a continuous map $f : A \to B$, say, with $f(x) = y$. One can then extend f to a continuous map $\tilde{f} : [0,1] \to [0,1]$ (or $\tilde{f} : \mathbb{R} \to \mathbb{R}$), whose code will still be in V.

Remark 20. If cov(nowhere dense) $= \mathfrak{c}$ for the unit interval I, then Remark 4 shows that $\mathfrak{tf}(I) = \mathfrak{c}$. Suppose that cov(nowhere dense) $= \kappa < \mathfrak{c}$, for I, then we can cover I by κ many Cantor sets $\{C_\alpha\}_{\alpha<\kappa}$ in such a way that for every two reals x and y there exists an α such that $x, y \in C_\alpha$. For every α we have a transitive family of continuous functions $\mathcal{F}_\alpha$ on C_α such that $|\mathcal{F}_\alpha| = \mathfrak{tf}(C)$. We can extend every $f \in \mathcal{F}_\alpha$ to a continuous self map $\tilde{f}$ of I. So $\mathcal{F} = \{\tilde{f}\colon$ there is an $\alpha < \kappa$ and $f \in \mathcal{F}_\alpha\}$ is a transitive set of continuous functions on I, and its cardinality is less than or equal to $\kappa \times \mathfrak{tf}(C) = \mathfrak{tf}(C)$.

So if we can cover the unit interval with less than $\mathfrak{c}$ many nowhere dense sets we have $\mathfrak{tf}(I) \leqslant \mathfrak{tf}(C)$.

4. The cardinal $\mathfrak{tf}$ and side-by-side Sacks forcing

In this paper we showed that after adding $\aleph_2$ many Sacks reals iteratively to a model of ZFC + CH we end up with a model of $\mathfrak{tf} < \mathfrak{c}$. Now consider $\mathbb{PS}(\kappa)$, the poset for adding κ many Sacks reals side-by-side (see [1]). We have that $\mathbb{PS}(\kappa)$ has the $(2^{\aleph_0})^+$-chain condition and preserves $\aleph_1$. Suppose that $\kappa \geqslant \aleph_1$ and $\mathrm{cf}(\kappa) \geqslant \aleph_1$. If V is a model of CH and G is $\mathbb{PS}(\kappa)$-generic over V, we have in $V[G]$ that $2^{\aleph_0} = \kappa$ and all cardinals are preserved.

A natural question would be if we get a model of $\mathfrak{tf} < \mathfrak{c}$ when we add $\aleph_2$ many Sacks reals side-by-side to a model of ZFC+CH. The answer to this question is in the negative.

Suppose that V is a model of ZFC. Consider the poset $\mathbb{P} = \mathbb{PS}(\{1,2,3,4\})$ that adds four Sacks reals side-by-side to the model V. We define $\mathbb{P}_1$ to be the p.o.-set $\mathbb{PS}(\{1,2\})$ and $\mathbb{P}_2$ to be the p.o.-set $\mathbb{PS}(\{3,4\})$. Suppose G is $\mathbb{P}$ generic over V then $G_{12} = G \restriction \{1,2\}$ is $\mathbb{P}_1$ generic and $G_{34} = G \restriction \{3,4\}$ is $\mathbb{P}_2$ generic over V. The following holds.

Lemma 21. *In $V[G]$ we have $V[G_{12}] \cap V[G_{34}] = V$.*

Proof. Suppose that $\dot{X}$ is a $\mathbb{P}$ name and q an element of $\mathbb{P}$ such that $q \Vdash$ "$\dot{X} \in V[G_{12}] \cap V[G_{34}]$". So there exists a $\mathbb{P}_1$ name $\dot{Y}$ and a $\mathbb{P}_2$ name $\dot{Z}$ such that $q \Vdash$ "$\dot{X} = \dot{Y} = \dot{Z}$". Aiming for a contradiction assume $\dot{X}$ is a name for an object not in V. There exists a $n \in \omega$ such that q does not decide $n \in \dot{X}$. Now we have $q_1 = q \restriction \{1,2\}$ does not decide $n \in \dot{Y}$, and $q_2 = q \restriction \{3,4\}$ does not decide $n \in \dot{Z}$. So we can find in $\mathbb{P}_1$ a $r \geqslant q_1$ such that $r \Vdash$ "$n \in \dot{Y}$" and in $\mathbb{P}_2$ a $t \geqslant q_2$ such that $t \Vdash$ "$n \notin \dot{Z}$". This gives the contradiction we are looking for because $r \cup t \Vdash$ "$\dot{Y} \neq \dot{Z}$" and $r \cup t \geqslant q$. So $\dot{X}$ must be a name of an element in V. □

Now we can prove that adding $\aleph_2$ many Sacks reals to a model of ZFC+CH we do not produce a model of $\mathfrak{tf} < \mathfrak{c}$.

Theorem 22. *Suppose $V \models$ CH and G is a $\mathbb{PS}(\kappa)$-generic filter over V, where $\kappa \geqslant \aleph_1$ and $\mathrm{cf}(\kappa) \geqslant \aleph_1$, then $V[G] \models \mathfrak{tf} = \mathfrak{c}$.*

Proof. For every $\alpha < \beta < \kappa$ we have that there exists a function $f_{\alpha,\beta} \in V[G \restriction \{\alpha,\beta\}]$ mapping s_α to s_β or vice versa. This function $f_{\alpha,\beta}$ is not a member of V for the obvious reason that assuming that $f_{\alpha,\beta}$ maps s_α to s_β we get $s_\beta \in V[G \restriction \{\alpha\}]$, which, of course, is false. Using Lemma 21 and the fact that $2\kappa = \kappa$ we see that the size of $\mathfrak{tf}$ is at least κ, because $f_{2\alpha,2\alpha+1} \neq f_{2\beta,2\beta+1}$ for every $\alpha \neq \beta$. By Theorem 3 we are done. □

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