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MORE REMARKS ON SOUSLIN PROPERTIES AND TREE TOPOLOGIES

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We investigate separation properties of ω_1-trees. We show that the property γ of Devlin and Shelah is equivalent to hereditary collectionwise normality. We show that monotone normality and divisibility are both equivalent to orderability. Finally we show that Souslin trees are examples of trees with property γ which are not retractable.

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ω_1-tree	hereditary collectionwise normal	divisible
Souslin tree	monotonically normal	orderable
property γ	property δ	retractable

0. Introduction

In this note we continue the investigation of separation properties in tree spaces which was started in [4] and [7]. First we show that trees with property γ are hereditarily collectionwise normal, improving [7; Theorem 2.1]. Next we consider some separation properties which every locally compact zero-dimensional Linearly Ordered Topological Space has, namely monotone normality, divisibility and retractability. We show that the first two are equivalent to orderability for ω_1-trees. As a byproduct we see that monotonically normal trees are retractable, it is unknown whether the converse holds. Finally we show that Souslin trees are not retractable, thus showing that HCWN trees need not be retractable.

1. Definitions

A tree is a poset $T=\langle T, <_T\rangle$ such that for all $x\in T$, $\hat{x}=\{y\in T\mid y<_T x\}$ is well ordered by $<_T$. The order type of $\hat{x}$ is denoted by $\mathrm{ht}(x)$, the height of x. $T_\alpha=\{x\in T\mid \mathrm{ht}(x)=\alpha\}$ is the α-th level of T. $T\restriction\alpha=\{x\in T\mid \mathrm{ht}(x)<\alpha\}$. If C is a set of ordinals, then $T\restriction C=\{x\in T\mid \mathrm{ht}(x)\in C\}$. A branch is a maximal chain. An α-branch is a branch of length α. An antichain is a subset of pairwise incomparable elements.

$\Lambda = \{\alpha \in \omega_1 \mid \alpha$ is a limit$\}$. For $x \in T$, $T^x = \{y \in T \mid x <_T y\}$. T is an ω_1-tree iff

(i) $T_{\omega_1} = \emptyset$.

(ii) $\forall \alpha \in \omega_1: 0 < |T_\alpha| \leq \omega_0$,

(iii) $\forall \alpha \in \beta \in \omega_1\ \forall x \in T_\alpha\ \exists y_1, y_2 \in T_\beta: (y_1 \neq y_2 \wedge x <_T y_1 \wedge x <_T y_2)$,

(iv) $\forall \alpha \in \omega_1\ \forall x, y \in T_\alpha (\lim(\alpha) \to (x = y \Leftrightarrow \hat{x} = \hat{y}))$.

We assume in addition that T_0 consists of one point $\mathbf{0}$, the root of the tree.

The tree topology on T is defined by taking the following collection as an open basis:

$$\{\{\mathbf{0}\}\} \cup \{(y, x] \mid y <_T x,\ y, x \in T\} \quad \text{where } (y, x] = \{z \mid y <_T z \leq_T x\}.$$

With this topology T is first-countable and locally compact.

Clause (iv) in the definition of ω_1-trees ensures that these trees are Hausdorff and zero-dimensional.

An ω_1-tree T is called Aronszajn iff it has no uncountable branches and Souslin iff it has no uncountable antichains. T is said to have property γ [4] iff the following holds:

If $A \subseteq T$ is an antichain, then there are a cub set $C \subset \omega_1$ and an open set $U \subset T$ such that $A \subset U \subset \bar{U} \subset T \backslash (T \restriction C)$.

T is said to have property δ iff there is a function $f: T \restriction \Lambda \to T$ such that

(i) $\forall x \in T \restriction \Lambda\ f(x) <_T x$

(ii) $\forall x, y \in T \restriction \Lambda$ if $[f(x), x] \cap [f(y), y] \neq \emptyset$, then $x \leq y$ or $y \leq x$.

For standard topological notions we refer to [6], additional definitions will be given when needed.

2. Normality properties

In [7] Fleissner showed that an ω_1-tree is collectionwise normal iff it has property γ. Modifying his proof we get the following result:

2.1. Theorem. *Let T be an ω_1-tree. Then*
T has property $\gamma \Leftrightarrow T$ is hereditarily collectionwise normal

Proof. Only '$\Rightarrow$' needs proof. So assume T is collectionwise normal. Let $\mathscr{F} = \{F_i \mid i \in I\}$ be a collection of subsets of T such that $\forall i \in I: F_i \cap \overline{\bigcup_{j \neq i} F_j} = \emptyset$. We have to find a family $\{U_i \mid i \in I\}$ of disjoint open sets s.t. $\forall i \in I: F_i \subset U_i$.

For $a \in \bigcup \mathscr{F}$ we pick $i(a) \in I$ s.t. $a \in F_{i(a)}$ and we put

$$B(a) = \{x \mid x \text{ is minimal in } T^a \cap \bigcup_{j \neq i(a)} F_j\}.$$

We define, for all $\eta \in \omega_1$, $A_\eta \subset \bigcup \mathscr{F}$ as follows:

– $A_0 = \{a \mid a$ is minimal in $\bigcup \mathscr{F}\}$.

– $A_{\eta+1} = \bigcup \{B(a) \mid a \in A_\eta\}$.

– If η is a limit put $D_\eta = \{d \in \bigcup \mathscr{F} \mid \forall \nu \in \eta: A_\nu \cap \hat{d} \neq \emptyset\}$ and let $A_\eta = \{a \mid a$ is minimal in $D_\eta\}$.

Furthermore for all $a \in A_\eta$ choose $x_a <_T a$ in such a way that $\{(x_a, a]\}_{a\in A_\eta}$ is discrete (by CWN) and

$$\forall a \in A_\eta: \quad (x_a, a] \cap \bigcup_{j \neq i(a)} F_j = \emptyset.$$

Note that $(x_a, a] \cap \bigcup_{\nu\in\eta} A_\nu = \emptyset$, for if $p \in (x_a, a] \cap A_\nu$ and $q \in (x_a, a] \cap A_{\nu+1}$, then $i(p) \neq i(q)$.

Put $A = \bigcup_{\eta\in\omega_1} A_\eta$ and define, for all $a \in A$, $X(a)$ as follows:

– If $a \in A_\eta$ and η is a successor or 0 put

$$X(a) = T^a \setminus \bigcup \{T^b \mid b \in B(a)\}.$$

– If $a \in A_\eta$ and η is a limit put

$$X(a) = (x_a, a] \cup T^a \setminus \bigcup \{T^b \mid b \in B(a)\}.$$

It is easy to see that each $X(a)$ is clopen in T.

Next we show that $X(a) \cap X(b) = \emptyset$ if $a \neq b$.

If $a, b \in A_\eta$ for some η this follows from the fact that A_η is an antichain and that – in case η is a limit – $(x_a, a] \cap (x_b, b] = \emptyset$.

If $a \in A_\nu$, $b \in A_\eta$ with $\nu \in \eta$, then let b' be the point of A_ν below b.

If $b' = a$, then $X(b) \subset T^c$ for some $c \in B(a)$; if $b' \neq a$, then $X(b) \subset T^{b'}$. In either case $X(b) \cap X(a) = \emptyset$.

Furthermore $\bigcup \mathscr{F} \subset \bigcup_{a\in A} X(a) \cup A_0$.

For take $x \in \bigcup \mathscr{F}$. If $\hat{x} \cap A = \emptyset$, then x must be minimal in $\bigcup F$ so $x \in A_0$. If $\hat{x} \cap A \neq \emptyset$, then let η be the first ordinal for which $\hat{x} \cap A_\eta = \emptyset$. If $\eta = \nu + 1$, then $x \in X(a)$ where a is the point in $A_\nu \cap \hat{x}$, if η is a limit, then $x \in D_\eta$ but since $\hat{x} \cap A_\eta = \emptyset$, we have $x \in A_\eta$ so $x \in X(x)$.

Finally, for each $a \in A$, $\overline{F_{i(a)}} \cap B(a) = \emptyset$, so we can find disjoint open sets U_a, $V_a \subset X(a)$ around $F_{i(a)} \cap X(a)$ and $B(a)$, respectively, furthermore we can find disjoint open sets around the points of $B(a)$, contained in V_a. We can also find disjoint open sets around the points of A_0. If we now form appropriate unions we get the desired collection of open sets separating $\mathscr{F}$. □

We remark that virtually the same proof shows that normality and hereditary normality are equivalent for ω_1-trees. Next we consider some separation properties which are possessed by linearly ordered topological spaces and which imply hereditary collectionwise normality, namely monotone normality and divisibility. It turns out that these properties are equivalent to orderability in ω_1-trees. We start with the definitions.

2.2. Definition. Let X be a topological space.

(a) X is called monotonically normal [8] iff to each pair $\langle U, x\rangle$ with $U \subset X$ open and $x \in U$ one can assign an open set U_x such that (i) $x \in U_x \subset U$ and (ii) if $U_x \cap V_y \neq \emptyset$, then $x \in V$ or $y \in U$. (This is in fact a characterization from [1]).

(b) X is called halvable (for lack of a better name) iff for each neighborhood assignment $x \to U_x$ there is another one $x \to V_x$ such that if $V_x \cap V_y \neq \emptyset$, then $x \in U_y \vee y \in U_x$. Halvability is a property of monotonically normal spaces which in some proofs is the only thing used. For instance, the proof that monotonically normal spaces are hereditarily collectionwise normal uses only halvability. Furthermore all countable regular spaces are easily seen to be halvable, so halvable spaces need not be monotonically normal. These facts were observed by I. Juhász.

(c) X is called divisible iff the collection of all neighborhoods of the diagonal $\Delta(X)$ in $X \times X$ is a uniformity or equivalently if for each open set $U \supset \Delta(X)$ there exists an open set $V \supset \Delta \supset (X)$ s.t. $V \circ V \subset U$. The name divisible appears in [2] and [3], the name strongly collectionwise normal in [9], however these spaces need not be strongly normal, which is why we adopt the name divisible.

Using the usual Pressing Down Lemma it is easy to prove the following.

Lemma (Pressing Down Lemma for ω_1-trees). *Let T be an ω_1-tree and let $A \subseteq T$ be a set which meets stationary many levels. Let $f: A \to T$ be a function s.t. $f(x) <_T x$ for all $x \in A$. Then f is constant on a set which meets stationary many levels.*

We now come to our orderability theorem for ω_1-trees.

Theorem 2.2. *The following are equivalent for an ω_1-tree T:*
(a) *T is monotonically normal.*
(b) *T is halvable.*
(c) *T is divisible.*
(d) *T has property δ.*
(e) *T is orderable.*

Proof. (a)$\Rightarrow$(b). See the definition

(b)$\Rightarrow$(d). Consider the assignment $x \to [\mathbf{0}, x]$. Let $x \to V_x$ be as in the definition. Define $f: T \restriction \Lambda \to T$ s.t. $\forall x \in T \restriction \Lambda\ f(x) <_T x$ and $[f(x), x] \subset V_x$. Then f is as required.

(c)$\Rightarrow$(d). Let $U = \bigcup_{x \in T} [\mathbf{0}, x]^2$ and let $V \supset \Delta T$ be open such that $V \circ V \subset U$ and $V = V^{-1}$. For all $x \in T \restriction \Lambda$ take $f(x) <_T x$ such that $[f(x), x]^2 \subset V$. Assume $[f(x), x] \cap [f(y), y] \neq \emptyset$ and take z in the intersection. Then $\langle x, z \rangle \in V$ and $\langle z, y \rangle \in V$ so $\langle x, y \rangle \in U$, hence $\{x, y\} \subset [\mathbf{0}, u]$ for some $u \in T$. But then $x \leqslant_T y$ or $y \leqslant_T x$.

(e)$\Rightarrow$(a) and (e)$\Rightarrow$(c) are well known, so we now prove:

(d)$\Rightarrow$(e) Let $f: T \restriction \Lambda \to T$ witness property δ, we can assume that $f(x) \notin T \restriction \Lambda$ for all x. From now on we let z_x denote $f(x)$. Let

$$A = \{z_x \mid x \in T \restriction \Lambda\}, \qquad P_z = \{x \mid z_x = z\}, \quad z \in A,$$

note that $x, y \in P_z \Rightarrow x \leqslant_T y$ or $y \leqslant_T x$,

$$Q_z = \bigcup_{x \in P_z} [z, x], \qquad B = \{z \mid P_z \text{ meets stationary many levels}\}.$$

Note that Q_z is linearly ordered since P_z is.

Claim. If $z_1, z_2 \in B$, then $Q_{z_1} \cap Q_{z_2} = \emptyset \vee Q_{z_1} \subset Q_{z_2} \vee Q_{z_2} \subset Q_{z_1}$.

Proof. Suppose $Q_{z_1} \not\subset Q_{z_2}$ and $Q_{z_2} \not\subset Q_{z_1}$. Then $\exists x \in Q_{z_1}, \exists y \in Q_{z_1}$ s.t. x and y are incomparable. For if not we have, say, $z_1 \leq_T z_2$. Take $y \in Q_{z_2}$ and choose $x \in Q_{z_1}$ s.t. $\mathrm{ht}(y) < \mathrm{ht}(x)$. It then follows that $y <_T x$ so $y \in [z_2, x] \subset [z_1, x] \subset Q_{z_1}$. Hence $Q_{z_2} \subset Q_{z_1}$. Hence $Q_{z_2} \subset Q_{z_1}$, a contradiction. So pick $x \in Q_{z_1}$ and $y \in Q_{z_2}$ s.t. x and y are incomparable, take $u \in P_{z_1}$ and $v \in P_{z_2}$ s.t. $x \leq_T u$ and $y \leq_T v$. Then u and v are incomparable so $[z_1, u] \cap [z_2, v] = \emptyset$ and hence $Q_{z_1} \cap Q_{z_2} = \emptyset$, which completes the proof of the claim.

Now let $z \in B$ and consider $\{u \in B \mid Q_z \subset Q_u\}$. Let z_0 be its minimum. Then $Q_z \subset Q_{z_0}$ and Q_{z_0} is maximal in $\{Q_u \mid u \in B\}$. Put $C = \{z \in B \mid Q_z \text{ is maximal}\}$. Then for $z_1, z_2 \in C$ we have $z_1 \neq z_2 \Rightarrow Q_{z_1} \cap Q_{z_2} = \emptyset$ and we have $Q = \bigcup_{z \in B} Q_z = \bigcup_{z \in C} Q_z$. Now each Q_z is clopen in T since $z \notin T \restriction \Lambda$ so $Q = \bigoplus_{z \in C} Q_z$ (topological sum). Q is open since Q_z is open. Q is closed: Let $x \in T \setminus Q$ be non-isolated i.e. $x \in T \restriction \Lambda$. Then $[z_x, x] \cap Q = \emptyset$. If not, then $[z_x, x] \cap Q_z \neq \emptyset$ for some z. Pick $y \in P_z$ s.t. $\mathrm{ht}(x) < \mathrm{ht}(y)$. Then $[z_x, x] \cap [z, y] \neq \emptyset$ and hence $x <_T y$. But then $x \in [z, y] \subset Q_z$, contradiction. So Q is clopen.

Next suppose $S = \{\mathrm{ht}(x) \mid x \in T \setminus Q\}$ is stationary. By the P.D.L. for trees there is a $z \in T$ and a set $K \subset (T \setminus Q) \cap (T \restriction \Lambda)$ such that $\{\mathrm{ht}(x) \mid x \in K\}$ is stationary and $\forall x \in K : z_x = z$. But then $z \in B$ since $K \subset P_z$ and hence $K \subset Q_z \subset Q$ contradiction. Let $M \subset \omega_1$ be c.u.b. s.t. $(T \setminus Q) \cap (T \restriction M) = \emptyset$ and let $\{m_\alpha \mid \alpha \in \omega_1\}$ be its monotone enumeration. Put

$$L_\alpha = \{x \in T \setminus Q \mid m_\alpha < \mathrm{ht}(x) < m_{\alpha+1}\}, \quad \alpha \in \omega_1.$$

Each L_α is countable and metrizable, so $T \setminus Q = \bigoplus_{\alpha \in \omega_1} L_\alpha$ is metrizable and strongly zerodimensional and hence orderable. Now $T = (T \setminus Q) \oplus \bigoplus_{z \in C} Q_z$ can be ordered as follows: Order the Q_z's two by two in type $\omega_1^* + \omega_1$, i.e. as (—][—) but keep one Q_{z_0} aside. Order the union of the paired Q_z's in type $\omega_1 \times (\omega_1^* + \omega_1)$ lexicographically and put Q_{z_0} at the beginning giving the following picture:

[—) (—][—) (—][—) $\cdots$ (—][—) (—][—) $\cdots$

Now order $T \setminus Q$ in some way and place it at the beginning or somewhere in the middle so as not to create any pseudogaps. $\square$

Remark. The P.D.L. for trees can be used to show two more things:

(1) No Aronszajn tree has property δ. For let $f : T \to T \restriction \Lambda$ be a function s.t. $\forall x \in T : \Lambda : f(x) <_T x$. There is an uncountable set on which f is constant. This set is not linearly ordered by $<_T$. So we find incomparable x and y such that $[f(x), x] \cap [f(y), y] \neq \emptyset$.

(2) No ω_1-tree is metalindelöf (=every open cover has a point-countable refinement). For let $\mathcal{V}$ be an open refinement of $\{[\mathbf{0}, x]\}_{x \in T}$. Let $f : T \restriction \Lambda \to T$ be a function such that $\forall x \in T \restriction \Lambda : f(x) <_T x$ and $[f(x), x] \subset$ some $V \in \mathcal{V}$. Again we find an uncountable set $A \subseteq T \restriction \Lambda$ and a point $z \in T$ s.t. $f(x) = z$ for all $x \in A$. But then z is contained in uncountably many elements of $\mathcal{V}$ i.e. $\mathcal{V}$ is not point-countable.

3. Retractability

We start with the definition.

3.1. Definition. A topological space X is called retractable iff each closed subset of X is a retract of X, i.e., for each closed set $A \subset X$ there is a continuous map

$$r : X \to A \quad \text{s.t.} \quad r \restriction A = \mathrm{id}_A.$$

See [5] for more information. In [5] it is shown that retractable spaces are hereditarily collectionwise normal and that locally compact zero-dimensional linearly ordered topological spaces are retractable. So, by Theorem 2.2, trees with property δ are retractable. Two questions now arise naturally: (1) Must retractable ω_1-trees have property δ, and (2) must ω_1-trees with property γ be retractable. We were unable to answer question (1), but we shall provide a negative answer to question (2). In fact we shall show that if T is a Souslin tree, then $T \restriction \Lambda$ is not a retract of T.

First we reduce the problem a little bit. For convenience we assume in this section that 0 is also a limit ordinal.

3.2. Lemma. *Assume $f : T \to T \restriction \Lambda$ is a retraction, then we can find another retraction $r : T \to T \restriction \Lambda$ with the following property:*

If $x \in T \backslash (T \restriction \Lambda)$, then

(1) $r(x) <_T x$, *or*

(2) $x <_T r(x)$, $\mathrm{ht}(r(x)) = \mathrm{ht}(x) + \omega$ *and* $x \leqslant_T y \leqslant_T r(x) \to r(y) = r(x)$.

Such a retraction will be called a nice retraction.

Proof. We put $\Lambda^2 = \{\alpha \in \omega_1 \mid \alpha \text{ is a limit of limits}\}$. If $\mathrm{ht}(p) \in \Lambda \backslash \Lambda^2$, then p is isolated in $T \restriction \Lambda$ so we can define

$$x_p = \min\{x \in \hat{p} \mid f[[x, p]] = \{p\}\}.$$

Now define $r : T \to T \restriction \Lambda$ as follows:

- If $p \in T \restriction \Lambda$ put $r(p) = p = f(p)$.
- If $x \in [x_p, p]$ for some p put $r(x) = p = f(x)$.
- If $x \notin (T \restriction \Lambda) \cup \bigcup_p [x_p, p]$ put $r(x) = \max(\hat{x} \cap T \restriction \Lambda)$.

Obviously r is a map satisfying (1) and (2), so it remains to show that r is continuous. Take $q \in T$.

If $\mathrm{ht}(q)$ is a successor or 0, then q is isolated and hence r is continuous at q.

If $\mathrm{ht}(q) \in \Lambda \backslash \Lambda^2$, then r is constant on the neighborhood $[x_q, q]$ of q, hence r is continuous at q.

Finally assume $\mathrm{ht}(q) \in \Lambda^2$ and let $y <_T q$. By continuity of f there is a $z <_T q$ such that $f[(z, q]] \subset (y, q]$, we can assume that $y \leqslant_T z$ and that $z = x_p$ for some $p \in T \restriction \Lambda \cap \hat{q}$. Take $x \in (z, q)$. If $x \in (x_s, s]$ for some s, then $r(x) = s = f(x)$, so $r(x) \in (y, q]$.

If $x \notin (x_s, s]$ for all s, then $p <_T x$, hence by definition of $r: p \leqslant_T r(x) < x < q$. So $r[(z, q]] \subset (y, q]$ and we can conclude that r is continuous at q. □

Now we prove the main result of this section.

3.3. Theorem. *Let T be an ω_1-tree s.t. $T \restriction \Lambda$ is a retract of T. Then T contains an uncountable antichain.*

Proof. By the lemma let $r: T \to T \restriction \Lambda$ be a nice retraction. For $q \in T \restriction (\Lambda^2)$ put

$$x_q = \min\{x \in \hat{q} \mid r[[x, q]] \subset [\mathbf{0}, q]\}.$$

If $x_q \in T \restriction \Lambda$, then $\exists y <_T x_q: r[[y, x_q]] \subseteq [\mathbf{0}, x_q]$, contradicting the choice of x_q.

Consider $r(x_q^-)$. We cannot have $r(x_q^-) < x_q$ for in that case $r[[x_q^-, q]] \subset [\mathbf{0}, q]$, and if $x_q < r(x_q^-)$, then because r is nice, $r(x_q) = r(x_q^-)$, so again $r[[x_q^-, q]] \subset [\mathbf{0}, q]$ which contradicts the choice of x_q. We conclude therefore that $r(x_q^-)$ and x_q are incomparable.

Now put $K = \{x_q \mid q \in T \restriction (\Lambda^2)\}$. K has the following two properties:

(α) For all $t \in T$ $\exists x \in K: t <_T x$.

Take $t \in T$ and fix a point p above t such that $\mathrm{ht}(p) = \mathrm{ht}(t) + \omega$. Pick $x \in [t, p)$s.t. $r[[x, p]] = \{p\}$, let x^+ be a successor of x not below p and take $q \in T \restriction (\Lambda^2)$ above x^+. Then $x \in \hat{q}$ but $r(x) = p \in [\mathbf{0}, q]$, so $t \leqslant x < x_q$.

(β) For all $t \in T$, $\hat{t} \cap K$ is finite.

Suppose to the contrary that for some $t \in T$, $\hat{t} \cap K$ is infinite and let $\{x_i \mid i \in \omega\}$ be its initial segment of length ω. Note that

$$x_0 \leqslant x_1^- < x_1 \leqslant x_2^- \leqslant x_3^- \cdots < t.$$

Let $x = \sup_{n \in \omega} x_n = \sup_{n \in \omega} x_n^-$. Since, for all n, $r(x_n^-)$ and x_n are incomparable we have that $r(x_n^-) \notin [\mathbf{0}, x]$ for all n. On the other hand $x_n^- \to x$, so $r(x_n^-) \to r(x) = x$, so $r(x_n^-) \in [\mathbf{0}, x]$ for at least one $n \in \omega$, which is a contradiction.

By (α) K is uncountable, by (β) $K = \bigcup_{i \in \omega} K_i$ where $K_i = \{x \in K \mid |\hat{x} \cap K| = i\}$, that is, K is the union of countably many antichains. One of these antichains is uncountable. □

3.4. Reformulation. *No Souslin tree T admits a retraction $r: T \to T \restriction \Lambda$.*

4. Remarks and questions

4.1. S. Todorcevic [10] showed that for an ω_1-tree having property δ is equivalent to being (isomorphic to) an initial segment of

$$T(\emptyset) = \{s \in \omega^{<\omega_1} \mid s(\alpha) \neq 0 \text{ for only finitely many } \alpha\}.$$

$T(\emptyset)$ is an example of a tree with property δ:

$$f(s) = s \restriction (\alpha(s)+1) \quad \text{where } \alpha(s) = \max\{\alpha \mid s(\alpha) \neq 0\},$$

defines a function $f: T(\emptyset) \restriction \Lambda \to T(\emptyset)$, which witnesses the fact that $T(\emptyset)$ has property δ.

4.2. Question. Is there (in some model of set theory) a retractable ω_1-tree which does not have property δ? Possible candidates are Aronszajn trees (they do not have property δ) or Kurepa trees (they have too many branches to be initial segments of $T(\emptyset)$).

4.3. Remark. In [11] it is shown than K_0- and K_1-trees are retractable and that for $n \geqslant 2$ a tree has property K_n iff the tree is collectionwise Hausdorff. See [5] for the definition of K_n-spaces. There it is shown that retractable spaces are K_0, that K_1-spaces are hereditarily collectionwise normal and for all n every K_n-space is a K_{n+1}-space. So by the results in this paper Souslin trees are examples of locally compact K_2-spaces which are not K_1. Their one-point compactifications are compact spaces with this property.

Note added in proof

Recently S. Todorčević showed that it is consistent relative to the existence of an at least inaccessible cardinal that all collectionwise Hausdorff (hence all retractable) trees are orderable.

References

[1] C.R. Borges, Four generalizations of stratifiable spaces, Proc. 3rd Prague Symp. (1971) 73–77.

[2] N. Bourbaki, Topologie Générale, Chap. 9, 2ième ed. (Hermann, Paris, 1958).

[3] Á. Császár, General Topology (Adam Hilger Ltd, Bristol, 1978).

[4] K.J. Devlin and S. Shelah, Souslin properties and tree topologies, Proc. London Math. Soc. 39 (1979) 237–252.

[5] E.K. van Douwen, Simultanuous extension of continuous functions, Thesis Free University Amsterdam (1975).

[6] R. Engelking, General Topology (PWN, Warszawa, 1977).

[7] W.G. Fleissner, Remarks on Souslin properties and tree topologies, Proc. AMS 80 (1980) 320–326.

[8] R.W. Heath, D.J. Lutzer and P.L. Zenor, Monotonically normal spaces, Trans. AMS 178 (1973) 481–494.

[9] H.L. Shapiro and F.A. Smith, Paracompactness in uniform spaces, Topology Proc. 3 (1978) 179–197.

[10] S. Todorčević, private communication.

[11] K.P. Hart, The K_n-property in ω_1-trees, to appear.