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Span, chainability and the continua $\mathbb{H}^*$ and $\mathbb{I}_u$

K.P. Hart *, B.J. van der Steeg

Faculty EEMCS, TU Delft, Postbus 5031, 2600 GA Delft, the Netherlands

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Dedicated to the memory of Zoli Balogh

Abstract

We show that the continua $\mathbb{I}_u$ and $\mathbb{H}^*$ are nonchainable and have span nonzero. Under CH this can be strengthened to surjective symmetric span nonzero.

We discuss the logical consequences of this.

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1. Introduction

Chainable (or arc-like) continua are ‘long and thin’; in an attempt to capture this idea in metric terms Lelek introduced, in [6], the notion of span. Chainable continua have span zero, which is useful in proving that certain continua are not chainable. The converse, a conjecture by Lelek in [7], is one of the main open problems in continuum theory today. While the particular value of the span of a continuum depends on the metric chosen, the distinction between span zero and span nonzero is a topological one. As chainability is a topological notion as well, Lelek’s theorem and conjecture are meaningful in the class of

* Corresponding author.
E-mail addresses: k.p.hart@ewi.tudelft.nl (K.P. Hart), berd@punch-basketball.nl (B.J. van der Steeg).
URL: http://aw.twi.tudelft.nl/~hart.

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all Hausdorff continua. We investigate the chainability and span of several continua that are closely connected to the Čech–Stone compactification of the real line.

2. Preliminaries

2.1. Various kinds of span

The kinds of span that we consider in this paper are, in the metric case, defined as suprema of distances between the diagonal of the continuum and certain subcontinua of the square. The following families of subcontinua feature in these definitions:

$S(X)$: the symmetric subcontinua of X^2, i.e., those that satisfy $Z = Z^{-1}$;
$\Sigma(X)$: the subcontinua of X^2 that satisfy $\pi_1[Z] = \pi_2[Z]$; and
$\Sigma_0(X)$: the subcontinua of X^2 that satisfy $\pi_2[Z] \subseteq \pi_1[Z]$.

Here, π_1 and π_2 are the projections onto the first and second coordinates, respectively. It is clear that $S(X) \subseteq \Sigma(X) \subseteq \Sigma_0(X)$ and hence that $s(X) \leqslant \sigma(X) \leqslant \sigma_0(X)$, where

(1) $s(X) = \sup\{d(\Delta(X), Z)\colon\ Z \in S(X)\}$;
(2) $\sigma(X) = \sup\{d(\Delta(X), Z)\colon\ Z \in \Sigma(X)\}$; and
(3) $\sigma_0(X) = \sup\{d(\Delta(X), Z)\colon\ Z \in \Sigma_0(X)\}$.

These numbers are, respectively, the *symmetric span*, the *span* and the *semi-span* of X.

If one uses, in each definition, only the continua Z with $\pi_1[Z] = X$ then one gets the *surjective symmetric span*, $s^*(X)$, the *surjective span*, $\sigma^*(X)$, and the *surjective semi-span*, $\sigma_0^*(X)$, of X, respectively. The following diagram shows the obvious relationships between the six kinds of span.

$$\begin{tikzcd} s(X) \arrow[r] & \sigma(X) \arrow[r] & \sigma_0(X) \\ s^*(X) \arrow[r] \arrow[u] & \sigma^*(X) \arrow[r] \arrow[u] & \sigma_0^*(X) \arrow[u] \end{tikzcd} \tag{1}$$

Topologically we can only distinguish between a span being zero or nonzero. A span is zero if and only if every continuum from its defining family intersects the diagonal. This defines span zero (or span nonzero) for the six possible types of span in general continua.

Below we will show that for the continua $\mathbb{H}^*$ and $\mathbb{I}_u$ all six kinds of span are nonzero. Diagram (1) shows that it will be most difficult to show that s^* is nonzero (or dually that it would be hardest to show that σ_0 is zero). Indeed, we will give successively more difficult proofs that the various spans are nonzero, where we traverse the diagram from top right to bottom left.

The need for these different proofs lies in their set-theoretic assumptions. We need nothing beyond ZFC to show that $\sigma^*(\mathbb{H}^*)$ and $\sigma(\mathbb{I}_u)$ are nonzero; to show that the other spans (in particular s^*) are nonzero we shall need the Continuum Hypothesis (CH).

2.2. Chainability

A continuum is *chainable* if every open cover of it has an open refinement that is a chain cover, where $\mathcal{C} = \{C_1, \dots, C_m\}$ *chain cover* if $C_i \cap C_j$ is nonempty if and only if $|i - j| \leqslant 1$.

One readily shows that every chainable continuum has span zero, whatever kind of span one uses. This follows from the fact that chainability is a hereditary property of continua and from the following theorem whose proof we give for completeness sake.

Theorem 2.1. *Every chainable continuum has surjective semi-span zero.*

Proof. Let X be a chainable continuum and let Z be a subcontinuum of X^2 that is disjoint from $\Delta(X)$. Let $\mathcal{U}$ be a finite open cover of X such that $U^2 \cap Z = \emptyset$ for all $U \in \mathcal{U}$. Next let $\{V_1, V_2, \dots, V_n\}$ be an open chain cover that refines $\mathcal{U}$. Define open sets O_1 and O_2 in X^2 by

$$O_1 = \bigcup\{V_i \times V_j \colon i < j\}, \qquad O_2 = \bigcup\{V_i \times V_j \colon i > j\}.$$

Then $Z \subset O_1 \cup O_2$ and $O_1 \cap O_2 = \emptyset$. As Z is connected, it is contained in one of O_1 or O_2, say $Z \subseteq O_2$. Then $\pi_1[Z] \subseteq \bigcup_{i<n} V_i$ and $\pi_2[Z] \subseteq \bigcup_{i>1} V_i$. This means that neither $\pi_1[Z]$ nor $\pi_2[Z]$ is equal to X. $\square$

2.3. The continua $\mathbb{I}_u$ and $\mathbb{H}^*$

In this paper we will be investigating the different kinds of span and the chainability of the continua $\mathbb{I}_u$ and $\mathbb{H}^*$. These two spaces are related to one another. Following [8,4], we will use the space $\mathbb{M} = \omega \times \mathbb{I}$ in our investigation of the spaces $\mathbb{I}_u$ and $\mathbb{H}^*$, where $\mathbb{I}$ denotes the unit interval $[0, 1]$.

The map $\pi : \mathbb{M} \to \omega$ given by $\pi(n, x) = n$ is perfect and monotone, as is its Čech–Stone extension $\beta\pi$. The preimage of an ultrafilter $u \in \omega^*$ is a continuum and denoted by $\mathbb{I}_u$.

Given any sequence $\langle x_n \rangle_{n\in\omega}$ in $\mathbb{I}$ and any $u \in \omega^*$ there is a unique point, denoted x_u, in $\mathbb{I}_u$ such that for every $\beta\mathbb{M}$-neighborhood O of x_u, the set $\{n \in \omega \colon (n, x_n) \in O\}$ is an element of u, i.e., x_u is the u-limit of the sequence $\langle (n, x_n) \rangle_{n\in\omega}$. These points form a dense set $\mathbb{C}_u$ of cut points of $\mathbb{I}_u$, for details see [4]. The set $\mathbb{C}_u$ is in fact the ultrapower of $\mathbb{I}$ by the ultrafilter u, i.e., the set ${}^{\omega}\mathbb{I}$ modulo the equivalence relation $x \sim_u y$ defined by $\{n \colon x_n = y_n\} \in u$.

The continuum $\mathbb{I}_u$ is irreducible between the points 0_u and 1_u (defined in the obvious way) and as it has a natural pre-order $\leqslant_u$ defined by $x \leqslant_u y$ iff every subcontinuum of $\mathbb{I}_u$ that contains 0_u and y also contains x. The equivalence classes under the equivalence relation "$x \leqslant_u y$ and $y \leqslant_u x$" are called layers and the set of layers is linearly ordered by $\leqslant_u$. The points of $\mathbb{C}_u$ provide one-point layers, the restriction of $\leqslant_u$ to this set coincides with the ultrapower order defined by $\{n \colon x_n \leqslant y_n\} \in u$. We shall freely use interval notation, allowing nontrivial layers as end points.

If $\langle x_n \rangle_{n\in\omega}$ is a strictly increasing sequence in $\mathbb{I}_u$ then its supremum L is a nontrivial layer. Because $\beta\mathbb{M} \setminus \mathbb{M}$ is an F-space the closure of $\{x_n \colon n \in \omega\}$ is homeomorphic to $\beta\omega$;

by upper semicontinuity the remainder (which is a copy of ω^*) must be contained in L. We call such a layer a countable-cofinality layer.

The continuum $\mathbb{H}^*$ is the remainder of the Čech–Stone compactification $\beta\mathbb{H}$, where $\mathbb{H}$ is the half line $[0,\infty)$. Let $q:\mathbb{M}\to\mathbb{H}$ be given by $q(n,x)=n+x$, then q is a perfect map and its Čech–Stone extension $\beta q:\beta\mathbb{M}\to\beta\mathbb{H}$ maps $\mathbb{M}^*$ onto $\mathbb{H}^*$. Again, for properties of $\mathbb{H}^*$ and its relation to $\mathbb{I}_u$ see [4].

3. The span of $\mathbb{H}^*$

In this section we show that the surjective (semi-)span of $\mathbb{H}^*$ is nonzero. The following theorem more than establishes this.

Theorem 3.1. *There exists a fixed-point free autohomeomorphism of* $\mathbb{H}^*$.

Proof. Let $f:\mathbb{H}\to\mathbb{H}$ be the map defined by $f:x\mapsto x+1$. It is clear that βf maps $\mathbb{H}^*$ onto $\mathbb{H}^*$. The restriction $f^*=\beta f\restriction\mathbb{H}^*$ is a fixed-point free autohomeomorphism of $\mathbb{H}^*$.

To see that f^* is an autohomeomorphism consider $g:\mathbb{H}\to\mathbb{H}$ defined by $g(x)=\max\{0,x-1\}$. From the fact that $f(g(x))=x$ and $g(f(x))=x$ for $x\geqslant 1$ it follows that $f^*\circ g^*$ and $g^*\circ f^*$ are the identity on $\mathbb{H}^*$.

That f is fixed-point free on $\mathbb{H}^*$ follows by considering the following closed cover $\{F_0,F_1,F_2,F_3\}$ of $\mathbb{H}$, defined by $F_i=\bigcup_n[2n+\frac{i}{2},2n+\frac{i+1}{2}]$. Observe that $f^*[F_i^*]=F_{i+2 \bmod 4}^*$ and that $F_i^*\cap F_{i+2 \bmod 4}^*$ is always empty, so that $f^*(x)\neq x$ for $x\in\mathbb{H}^*$. □

Corollary 3.2. $\sigma^*(\mathbb{H}^*)$ *is nonzero.*

Proof. The graph of f^* is a continuum in $\mathbb{H}^*\times\mathbb{H}^*$ that is disjoint from the diagonal and whose projection on each of the axes is $\mathbb{H}^*$. □

Later we shall see that under CH even $s^*(\mathbb{H}^*)$ is nonzero.

By Theorem 2.1 we also know that $\mathbb{H}^*$ is not chainable. The reader may enjoy showing that the four open sets U_0, U_1, U_2 and U_3 defined by

$$U_i=\bigcup_{n<\omega}(8n+2i,8n+2i+3)$$

induce an open cover of $\mathbb{H}^*$ without a chain refinement.

3.1. More fixed-point free homeomorphisms

We use the description of indecomposable subcontinua from [2] to show that many subcontinua of $\mathbb{H}^*$ have fixed-point free autohomeomorphisms.

We use the shift-map $\sigma:\omega\to\omega$, defined by $\sigma(n)=n+1$, and its extension to $\beta\omega$. We note that σ is an autohomeomorphism of ω^*. We also write $u+1$ for $\sigma(u)$ and $u-1$ for $\sigma^{-1}(u)$.

For $F \subseteq \omega^*$ we put $\mathbb{M}_F = \bigcup_{u \in F} \mathbb{I}_u$ and $C_F = \beta q[\mathbb{M}_F]$. We say that F is σ-invariant if $u+1, u-1 \in F$ whenever $u \in F$. Clearly then, if F is σ-invariant then $f^* \restriction C_F$ is an autohomeomorphism of C_F, where f^* is the autohomeomorphism of $\mathbb{H}^*$ defined in the proof of Theorem 3.1.

From [2] we quote the following: C_F is a subcontinuum whenever F is closed, σ-invariant and not the union of two disjoint proper closed σ-invariant subsets. In that case C_F is indecomposable if and only if F is dense-in-itself.

From [2] we also quote: if K is an indecomposable subcontinuum of $\mathbb{H}^*$ then there is a strictly increasing sequence $\langle a_n \rangle_n$ in $\mathbb{H}$ that diverges to ∞ and such that $K = q_a[C_F]$ for some closed dense-it-itself σ-invariant subset F of ω^* that is not the union of two disjoint proper closed σ-invariant subsets and where $q_a : \mathbb{H}^* \to \mathbb{H}^*$ is induced by the piecewise linear self-map of $\mathbb{H}$ that sends n to a_n.

We can combine all this into the following theorem.

Theorem 3.3. *Every indecomposable subcontinuum of $\mathbb{H}^*$ has a fixed-point free autohomeomorphism* (*and hence surjective span nonzero*).

4. The span of $\mathbb{I}_u$

In this section we show that $\mathbb{I}_u$ has span nonzero for any ultrafilter u; the next section will be devoted to the surjective versions of span.

The following theorem, akin to Theorem 3.1 and with a similar proof, provides a continuum witnessing that $\mathbb{I}_u$ has nonzero span.

Theorem 4.1. *Every countable-cofinality layer has a fixed-point free autohomeomorphism.*

This follows from Theorem 3.3 but for later use we give a direct construction, which establishes a bit more, namely that the interval $[0_u, L]$ has a fixed-point free continuous self-map.

Proof. We prove the theorem for one particular layer but the argument is easily adapted to the general case.

For $m \in \omega$ put $x_m = 1 - 2^{-m}$; then $\{x_m\}_{m<\omega}$ is a strictly increasing sequence in $\mathbb{I}$ that converges to 1 and with $x_0 = 0$. Let $x_{m,u}$ denote the point of $\mathbb{I}_u$ that corresponds to the constant sequence $\{x_m\}_{n\in\omega}$ in $\mathbb{I}$. Then $\{x_{m,u}\}_{m\in\omega}$ is a strictly increasing sequence in $\mathbb{I}_u$; let L denote the limit of this sequence, a nontrivial layer of $\mathbb{I}_u$.

We define a map $f : \mathbb{I}_u \to \mathbb{I}_u$ by defining it on $\mathbb{M}$, taking its Čech–Stone extension and restricting that to $\mathbb{I}_u$.

(1) Let $f \restriction \mathbb{I}_0$ be equal to the identity.
(2) For all $n \geqslant 1$ let $f \restriction \mathbb{I}_n$ be the piecewise linear map that maps (n, x_m) to (n, x_{m+1}) for all $m < n$ and the point $(n, 1)$ to itself.

Claim 1. *The Čech–Stone extension of the map f maps $[0_u, L]$ homeomorphically onto $[x_{1,u}, L]$.*

Proof. It is not hard to see that βf maps the interval $[x_{m,u}, x_{m+1,u}]$ of $\mathbb{I}_u$ homeomorphically onto $[x_{m+1,u}, x_{m+2,u}]$ for all $m \in \omega$. This implies that βf maps $[0_u, L)$ homeomorphically onto $[x_{1,u}, L)$. The fact that $[0_u, L] = \beta[0_u, L)$ now establishes the claim. □

We let h denote the restriction of βf to $[0_u, L]$. The fact that $[0_u, L] = \beta[0_u, L)$ also establishes the following claim.

Claim 2. *The restriction $h \restriction L$ maps L homeomorphically onto L.*

To see that h has no fixed points we argue as in the proof of Theorem 3.1.

For every m let a_m be the mid point of the interval (x_m, x_{m+1}). Note that the map f maps (n, a_m) onto the point (n, a_{m+1}) whenever $m < n$. Define the following closed subsets F_i for $i = 0$, 1, 2 and 3:

$$F_0 = \bigcup_n \left(\{n\} \times \bigcup_{m<n} [x_{2m}, a_{2m}]\right), \qquad F_2 = \bigcup_n \left(\{n\} \times \bigcup_{m<n} [x_{2m+1}, a_{2m+1}]\right),$$
$$F_1 = \bigcup_n \left(\{n\} \times \bigcup_{m<n} [a_{2m}, x_{2m+1}]\right), \qquad F_3 = \bigcup_n \left(\{n\} \times \bigcup_{m<n} [a_{2m+1}, x_{2m+2}]\right).$$

Note that the closure in $\beta\mathbb{M}$ of the union of the F_i's contains the interval $[0_u, L]$ of $\mathbb{I}_u$. Also note that the closed set F_i is mapped onto the closed set $F_{i+2 \bmod 4}$, so $f[F_i] \cap F_i = \emptyset$. As in the proof of Theorem 3.1 this implies that h has no fixed points. □

As before we get the following corollaries.

Corollary 4.2. *The surjective span of L is nonzero, hence $\sigma(\mathbb{I}_u)$ is nonzero.*

Corollary 4.3. *The surjective semi-span of $[0_u, L]$ is nonzero.*

It will be more difficult to prove the same for $\mathbb{I}_u$.

5. The surjective spans of $\mathbb{I}_u$ and $\mathbb{H}^*$

Using the map from the previous section and the retraction we get from the next theorem we will show that under CH there exists a fixed-point free continuous self map of $\mathbb{I}_u$; as the map is not onto this only implies that the surjective semi-span of $\mathbb{I}_u$ is nonzero. However, the special structure of $\mathbb{I}_u$ will allow us to build, using the graph of this map, a symmetric subcontinuum of $\mathbb{I}_u^2$ that will witness $s^*(\mathbb{I}_u) \neq 0$; it will then also be possible to show that $s^*(\mathbb{H}^*)$ is nonzero.

We retain the notation from the previous section but we write $a_m = x_{m,u}$ for ease of notation and we recall that layer L is the supremum, in $\mathbb{I}_u$, of the set $\{a_m\colon m \in \omega\}$. The following theorem is what makes the rest of this section work.

Theorem 5.1. (CH) *L is a retract of $[L, 1_u]$.*

Before we prove the theorem we give the promised consequences.

Theorem 5.2. (CH) *The continuum $\mathbb{I}_u$ does not have the fixed-point property.*

Proof. Let $h : [0_u, L] \to [0_u, L]$ be the map constructed in the proof of Theorem 4.1 and let $r : [L, 1_u] \to L$ be the retraction from Theorem 5.1. Extending r by the identity on $[0_u, L]$ yields a retraction r^* from $\mathbb{I}_u$ onto $[0_u, L]$. The composition $h \circ r^*$ is then a fixed-point free continuous self-map of $\mathbb{I}_u$. □

Corollary 5.3. (CH) *The surjective semi-span of $\mathbb{I}_u$ is nonzero.*

Proof. The graph of $h \circ r^*$ is a witness. □

We now show how to make $s^*(\mathbb{I}_u)$ nonzero.

Corollary 5.4. (CH) *The surjective symmetric span of $\mathbb{I}_u$ is nonzero.*

Proof. Let G be the graph of $h \circ r^*$. We complete G to symmetric continuum by adding the following continua: $\{1_u\} \times [0_u, L]$, $[h(0_u), 1_u] \times \{0_u\}$, G^{-1}, $[0_u, L] \times \{1_u\}$, and $\{0_u\} \times [h(0_u), 1_u]$. It is straightforward to check that the union Z is a continuum (each continuum meets its successor) that is symmetric and projects onto each axis. As none of the pieces intersects the diagonal we get a witness to $s^*(\mathbb{I}_u)$ being nonzero. □

Corollary 5.5. (CH) *The surjective symmetric span of $\mathbb{H}^*$ is nonzero.*

Proof. We begin by taking the graph F of the map f from Theorem 3.1 and its inverse F^{-1}; unfortunately the union $F \cup F^{-1}$ is not connected, as F and F^{-1} are disjoint. To connect them we take one ultrafilter u on ω and observe that the image $q[\mathbb{I}_u]$ connects the ultrafilters u and $u + 1$. The image $K = (q \times q)[Z]$, where Z is from the proof of Corollary 5.4 meets both F (in $(u, u+1)$) and F^{-1} (in $(u+1, u)$). The union $F \cup K \cup F^{-1}$ is a witness to $s^*(\mathbb{H}^*) \neq 0$. □

5.1. Proof of Theorem 5.1

We will construct the retraction by algebraic, rather than topological, means. Let $\mathcal{R}$ be the family of finite unions of closed intervals of $\mathbb{I}$ with rational endpoints. For every $f \in {}^{\omega}\mathcal{R}$ we define the closed subset A_f of $\mathbb{M}$ by

$$A_f = \bigcup_{n<\omega} \{n\} \times f(n).$$

These sets form a lattice base for the closed sets of $\mathbb{M}$, i.e., it is a base for the closed sets and closed under finite unions and intersections. It is an elementary exercise to show that disjoint closed sets in $\mathbb{M}$ can be separated by disjoint closed sets of the form A_f. This implies that the closures $\operatorname{cl} A_f$ form a lattice base for the closed sets of $\beta\mathbb{M}$. It follows that $\mathcal{B} = \{\operatorname{cl} A_f \cap L\colon\ f \in {}^{\omega}\mathcal{R}\}$ is a base for the closed sets of L and similarly that $\mathcal{C} = \{\operatorname{cl} A_f \cap [L, 1_u]\colon\ f \in {}^{\omega}\mathcal{R}\}$ is a base for $[L, 1_u]$.

Theorem 1.2 from [3] tells us that in order to construct a retraction from $[L, 1_u]$ onto L it suffices to construct a map $\varphi : \mathcal{B} \to \mathcal{C}$ that satisfies

(1) $\varphi(\emptyset) = \emptyset$, and if $F \neq \emptyset$ then $\varphi(F) \neq \emptyset$;
(2) if $F \cup G = L$ then $\varphi(F) \cup \varphi(G) = [L, 1_u]$;
(3) if $F_1 \cap \cdots \cap F_n = \emptyset$ then $\varphi(F_1) \cap \cdots \cap \varphi(F_n) = \emptyset$; and
(4) $\varphi(F) \cap L = F$.

The retraction $r : [L, 1_u] \to L$ is then defined by $r(x) =$ 'the unique point in $\bigcap\{F\colon\ x \in \varphi(F)\}$'. The first three conditions ensure that r is well-defined, continuous and onto; the last condition ensures that $r \restriction L$ is the identity.

There is a decreasing ω_1-sequence $\langle b_\alpha \rangle_{\alpha<\omega_1}$ of cut points in $\mathbb{I}_u$ such that $L = \bigcap_{m,\alpha}[a_m, b_\alpha]$: by [4, Lemma 10.1], such a sequence must have uncountable cofinality and by CH the only possible (minimal) length then is ω_1. For each α choose a sequence $\langle b_{\alpha,n} \rangle_{n\in\omega}$ in $\mathbb{I}$ such that $b_\alpha = b_{\alpha,u}$.

Again by CH we list ${}^{\omega}\mathcal{R}$ in an ω_1-sequence $\langle f_\alpha \rangle_{\alpha<\omega_1}$. We will assign to each f_α a $g_\alpha \in {}^{\omega}\mathcal{R}$ in such a way that $\operatorname{cl} A_{f_\alpha} \cap L \mapsto \operatorname{cl} A_{g_\alpha} \cap [L, 1_u]$ defines the desired map φ.

The assignment will be constructed in a recursion of length ω_1, where at stage α we assume the conditions (1)–(4) are satisfied for the A_{f_β} and A_{g_β} with $\beta < \alpha$ and choose g_α in such a way that they remain satisfied for $\beta \leqslant \alpha$. At every stage we will list α in an ω-sequence; this means that it suffices to consider the case $\alpha = \omega$ only.

We need a few lemmas that translate intersection properties in $\mathcal{B}$ and $\mathcal{C}$ to $\mathcal{R}$.

Lemma 5.6. $\operatorname{cl} A_f \cap L = \emptyset$ *if and only if there are m and α such that the set* $\{n\colon\ f(n) \cap [a_{m,n}, b_{\alpha,n}] = \emptyset\}$ *belongs to u.*

Proof. By compactness $\operatorname{cl} A_f \cap L = \emptyset$ if and only if there are m and α such that $\operatorname{cl} A_f \cap [a_m, b_\alpha] = \emptyset$ and the latter is equivalent to $\{n\colon\ f(n) \cap [a_{m,n}, b_{\alpha,n}] = \emptyset\} \in u$, again by compactness and the formula

$$\operatorname{cl} A_f \cap [a_m, b_\alpha] = \bigcap_{U\in u} \operatorname{cl}\biggl(\bigcup_{n\in U} \{n\} \times \bigl(f(n) \cap [a_{m,n}, b_{\alpha,n}]\bigr)\biggr). \qquad \square$$

Lemma 5.7. $\operatorname{cl} A_f \cap L = \operatorname{cl} A_g \cap L$ *if and only if there are m and α such that the set* $\{n\colon\ f(n) \cap [a_{m,n}, b_{\alpha,n}] = g(n) \cap [a_{m,n}, b_{\alpha,n}]\}$ *belongs to u.*

Proof. The 'if' part is clear. For the 'only if' part let D be the set of all mid points of all maximal intervals in $A_f \setminus A_g$; then $\operatorname{cl} D \subseteq \operatorname{cl} A_f \setminus \operatorname{cl} A_g$ and so $\operatorname{cl} D \cap L = \emptyset$. Observe that $D = A_h$ for some h, so there are m and α as in Lemma 5.6 for D. By convexity, for each n the interval $[a_{m,n}, b_{\alpha,n}]$ meets at most two of the maximal intervals in $f(n) \setminus g(n)$ —one, I_n, at the top and one, J_n, at the bottom. The two sequences $\langle i_n \rangle_{n\in\omega}$ (bottom points of the I_n) and $\langle j_n \rangle_{n\in\omega}$ (top points of the J_n) determine cut points i_u and j_u of $\mathbb{I}_u$, which cannot belong to L. Therefore we can enlarge m and α such that $\{n\colon i_n, j_n \notin [a_{m,n}, b_{\alpha,n}]\}$ is in u. A convexity argument will now establish that $\{n\colon (f(n) \setminus g(n)) \cap [a_{m,n}, b_{\alpha,n}] = \emptyset\}$ belongs to u. The same argument, interchanging f and g will yield our final m and α. □

Lemma 5.8. *$L \subset \operatorname{cl} A_f$ if and only if there are m and $\alpha < \omega_1$ such that the set $\{n\colon [a_{m,n}, b_{\alpha,n}] \subseteq f(n)\}$ belongs to u.*

Proof. Apply Lemma 5.7 to f and the constant function $n \mapsto \mathbb{I}$. □

Now we are ready to perform the construction of g_ω, given subsets $\{f_k\}_{k\leqslant\omega}$ and $\{g_k\}_{k<\omega}$ of ${}^\omega\mathcal{R}$ such that the map $\operatorname{cl} A_{f_k} \cap L \mapsto \operatorname{cl} A_{g_k} \cap [L, 1_u]$ $(k < \omega)$ satisfies the conditions (1)–(4) from our list.

The conditions that need to be met are

(a) $L \cap \operatorname{cl} A_{f_\omega} = L \cap \operatorname{cl} A_{g_\omega}$;
(b) if $L \subseteq \operatorname{cl} A_{f_k} \cup \operatorname{cl} A_{f_\omega}$ then $[L, 1_u] \subseteq \operatorname{cl} A_{g_k} \cup \operatorname{cl} A_{g_\omega}$; and
(c) if $F \subseteq \omega$ is finite and $L \cap \operatorname{cl} A_{f_\omega} \cap \bigcap_{l\in F} \operatorname{cl} A_{f_l} = \emptyset$ then $[L, 1_u] \cap \operatorname{cl} A_{g_\omega} \cap \bigcap_{l\in F} \operatorname{cl} A_{g_l} = \emptyset$.

The first condition takes care of (1) and (4) in our list, except possibly when $\operatorname{cl} A_{f_\omega} \cap L = \emptyset$ but in that case it suffices to let g_ω be the constant function $n \mapsto \emptyset$. The second and third condition ensure (2) and (3), respectively. There is one more condition that we need to keep the recursion alive; it is needed to take care of combinations of (b) and (c): if $L \subseteq \operatorname{cl} A_{f_k} \cup \operatorname{cl} A_{f_\omega}$ and $L \cap \operatorname{cl} A_{f_\omega} \cap \bigcap_{l\in F} \operatorname{cl} A_{f_l} = \emptyset$ then we must have room to be able to ensure that both $[L, 1_u] \subseteq \operatorname{cl} A_{g_k} \cup \operatorname{cl} A_{g_\omega}$ and $[L, 1_u] \cap \operatorname{cl} A_{g_\omega} \cap \bigcap_{l\in F} \operatorname{cl} A_{g_l} = \emptyset$. Note that the antecedent implies that, in the subspace L, the intersection $L \cap \bigcap_{l\in F} \operatorname{cl} A_{f_l}$ is contained in the interior of $L \cap \operatorname{cl} A_{f_k}$. A moment's reflection shows that we need

(d) if $L \cap \bigcap_{l\in F} \operatorname{cl} A_{f_l}$ is contained in $\operatorname{int}_L L \cap \operatorname{cl} A_{f_k}$ then $[L, 1_u] \cap \bigcap_{l\in F} \operatorname{cl} A_{g_l}$ is contained in $\operatorname{int}_{[L,1_u]}[L, 1_u] \cap \operatorname{cl} A_{f_k}$.

For every k as in (b) choose m_k and α_k as per Lemma 5.8 such that $U_k = \{n\colon [a_{m_k,n}, b_{\alpha_k,n}] \subseteq f_k(n) \cup f_\omega(n)\}$ belongs to u. Likewise, for every F as in (c) choose m_F and α_F as per Lemma 5.6 such that $U_F = \{n\colon [a_{m_F,n}, b_{\alpha_F,n}] \cap f_\omega(n) \cap \bigcap_{l\in F} f_l(n) = \emptyset\}$ belongs to u. And, finally, for every pair (F, k) as in (d) (with F finite but with $k \leqslant \omega$ in this case) choose $m_{F,k}$ and $\alpha_{F,k}$, and $U_{F,k} \in u$ such that for every $n \in U_{F,k}$ we have $[a_{m_{F,k},n}, b_{\alpha_{F,k},n}] \cap \bigcap_{l\in F} f_l(n) \subseteq \operatorname{int} f_k(n)$ and $[a_{m_{F,k},n}, 1] \cap \bigcap_{l\in F} g_l(n) \subseteq \operatorname{int} g_k(n)$ (the latter only if $k < \omega$ of course).

We fix an ordinal α larger than the α_k, α_F and $\alpha_{F,k}$ and use it instead in the definitions of the sets U_k, U_F and $U_{F,k}$—they will still belong to u. Next take a decreasing sequence $\langle V_p\rangle_{p\in\omega}$ of elements of u such that V_p is a subset of

- U_k whenever $k < p$;
- U_F whenever $F \subseteq p$; and
- $U_{F,k}$ whenever $F \subseteq p$ and $k < p$ or $k = \omega$.

In addition we can, and will, assume that whenever $F \subseteq p$ and $L \cap \bigcap_{l\in F} \operatorname{cl} A_f = \emptyset$ then $[b_\alpha, 1] \cap \bigcap_{l\in F} g_l(n) = \emptyset$—that this is possible follows from the assumption that (c) holds for $\max F$.

Now we are truly ready to define g_ω. If $n \notin V_0$ define $g_\omega(n) = \mathbb{I}$. In case $n \in V_p \setminus V_{p+1}$ observe first that if $k < p$ is as in (b) and $F \subseteq p$ is as in (c) then (F, k) is as in (d) so that certainly

$$[a_{m_{F,k}}, 1] \cap \bigcap_{l\in F} g_l(n) \subseteq \operatorname{int} g_k(n). \tag{$*$}$$

Define $g_\omega(n)$ as the union of $f_\omega(n) \cap [0, b_\alpha(n)]$ and an element $h(n)$ of $\mathcal{R}$ that is a subset of $[b_\alpha(n), 1]$ and satisfies

- $h(n) \cup g_k(n) \supseteq [b_\alpha(n), 1]$ whenever $k < p$ is as in (b);
- $h(n) \cap \bigcap_{l\in F} g_l(n) = \emptyset$ whenever $F \subseteq p$ is as in (c); and
- $h(n) \supseteq [b_{\alpha,n}, 1] \cap \bigcap_{l\in F} g_l(n)$ whenever (F, ω) is as in (d).

This is possible because of $(*)$ and because $\bigcap_{l\in F} g_l(n) \cap \bigcap_{l\in G} g_l(n) = \emptyset$ whenever F is as in (c) and (G, ω) is as in (d). This gives us just enough room to choose $h(n)$.

It is now routine to verify that all conditions on g_ω are met u-often: e.g., if $F \subseteq \omega$ is finite and $L \cap \operatorname{cl} A_{f_\omega} \cap \bigcap_{l\in F} \operatorname{cl} A_{f_l} = \emptyset$ then $[a_{m_{F,n}}, 1] \cap g_\omega(n) \cap \bigcap_{l\in F} g_l(n) = \emptyset$ for all $n \in V_p$, where $p = 1 + \max F$.

5.2. Further considerations

The proof in the previous section can be used to show that, under CH, all other layers of the continuum $\mathbb{I}_u$ are retracts of $\mathbb{I}_u$. If the layer is a point then this is clear. If the layer L is nontrivial then the cofinality of $[0_u, L)$ and the coinitiality of $(L, 1_u]$ are ω_1. It is then a matter of making the proof of Theorem 5.1 symmetric to get our retraction $r : \mathbb{I}_u \to L$. The details can be found in [9].

The fixed-point free homeomorphism $h : L \to L$ from Theorem 3.3 can then be used to construct another witness to $s^*(\mathbb{I}_u) \neq 0$, almost exactly as in the proof of Theorem 5.4.

6. Remarks

The results of this paper grew out of an attempt to find nonmetric counterexamples to Lelek's conjecture. The fairly easy proof, indicated after Corollary 3.2, that $\mathbb{H}^*$ is not chainable, which also works for layers of countable cofinality lead us to consider $\mathbb{I}_u$ as a possible candidate.

A secondary goal was to convert any nonmetric counterexample into a metric one by an application of the Löwenheim–Skolem theorem [5, Section 3.1], to its lattice of closed sets. This produces a countable sublattice with exactly the same (first-order) lattice-theoretic properties; its Wallman representation space, see [10], is a metrizable continuum with many properties in common with the starting space, e.g., covering dimension unicoherence, (hereditary) indecomposability, ..., see [9, Chapter 2], for a comprehensive list.

The results of this paper cast doubt of the possibility of adding (non)chainability and span (non)zero (of any kind) to this list. The reason for this is that the family $\mathcal{R}_u = \{\operatorname{cl} A_f \cap \mathbb{I}_u\colon f \in {}^{\omega}\mathcal{R}\}$ is isomorphic to the ultrapower of $\mathcal{R}$ (from the proof Theorem 5.1) by the ultrafilter u; this follows in essence from the equivalence of $\operatorname{cl} A_f \cap \mathbb{I}_u = \operatorname{cl} A_g \cap \mathbb{I}_u$ and $\{n\colon f(n) = g(n)\} \in u$. By the Łos Ultraproduct Theorem [5, Theorem 8.5.3], we see that $\mathcal{R}$ and $\mathcal{R}_u$ have the same first-order lattice theoretic properties yet their Wallman representations, $\mathbb{I}$ and $\mathbb{I}_u$, respectively, differ in chainability and in various kinds of span (all kinds if CH is assumed).

Chainability is a property that can be read off from a lattice base for the closed sets (or dually for the open sets): using compactness one readily shows that a continuum is chainable iff every basic open cover has a chain refinement from the base. Thus we deduce that chainability is not a first-order property of the lattice base.

For span (non)zero there are two possibilities: it cannot be read off from a base or, if it can be, it is not a first-order property of the lattice base.

7. Questions

The remarks in the previous section suggest lots of questions. We mention the more important ones.

Question 7.1. Is there a nonmetric counterexample to any one version of Lelek's conjecture?

It should be noted that, as mentioned in [1], H. Cook has shown that the dyadic solenoid has symmetric span zero.

In spite of the results on $\mathbb{I}$ and $\mathbb{I}_u$ it is still possible that the Löwenheim–Skolem method may convert a nonmetric counterexample into a metric one. The reason for this is that $\mathcal{R}_u$ is special base for the closed sets of $\mathbb{I}_u$ and not an elementary sublattice of its lattice of closed sets.

Question 7.2. If L is an elementary sublattice of the full lattice of closed sets of the continuum X, does its Wallman representation inherit (non)chainability and or span (non)zero from X?

Section 3.7 of [9] gives a positive answer for very special sublattices, but unfortunately except for span zero. Further, more specialized, questions can be found in that reference.

The corollaries in Section 5 were derived from Theorem 5.1, which needed CH in its proof. This clearly suggests the question whether a more insightful analysis of the structure of the $\mathbb{I}_u$ and the use of more intricate combinatorics will make the use of CH unnecessary.

Question 7.3. Can one show in ZFC only that all spans of $\mathbb{H}^*$ and $\mathbb{I}_u$ are nonzero?

It would already be of interest if one could find at least one u such that all spans of $\mathbb{I}_u$ are nonzero.

We have shown implicitly that the fixed-point property is like chainability and span zero in that $\mathbb{I}$ has it but $\mathbb{I}_u$ does not, at least under CH.

Question 7.4. Is there in ZFC at least one u such that $\mathbb{I}_u$ does not have the fixed-point property?

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