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A NEW SUBCONTINUUM OF $\beta\mathbb{R}\setminus\mathbb{R}$

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Abstract. We present a method for describing all indecomposable subcontinua of $\beta\mathbb{R}\setminus\mathbb{R}$. This method enables us to construct in ZFC a new subcontinuum of $\beta\mathbb{R}\setminus\mathbb{R}$.

We also show that the nontrivial layers of standard subcontinua can be described by our method. This allows us to construct a layer with a proper dense F_σ-subset and bring the number of (known) nonhomeomorphic subcontinua of $\beta\mathbb{R}\setminus\mathbb{R}$ to 14.

Introduction

The object of study of this paper is the Čech-Stone remainder $\mathbb{H}^*$ of the half line; the half line, denoted $\mathbb{H}$, is the subset $[0,\infty)$ of the real line $\mathbb{R}$. It is readily seen that $\beta\mathbb{R}\setminus\mathbb{R}$ is simply the topological sum of two copies of $\mathbb{H}^*$ so that no generality is lost by looking at just $\mathbb{H}^*$.

The half line is connected, so $\beta\mathbb{H}$ is a *continuum*, that is, a compact and connected space. It is not too hard to show that the remainder $\mathbb{H}^*$ is a continuum as well. Among the earliest known properties of $\mathbb{H}^*$ are its *hereditary unicoherence* (if two subcontinua meet then their intersection is connected as well) — established by Gillman and Henriksen in [6], and its *indecomposability* (it is not the union of two proper subcontinua) — established by Woods in [12] and Bellamy in [2].

In this paper we continue the investigation into the number of topologically different subcontinua of $\mathbb{H}^*$. So far through the efforts of van Douwen [11], Smith [10] and Zhu [13] nine different subcontinua of $\mathbb{H}^*$ have been discovered, in ZFC. We shall describe these continua briefly in Subsection 1.6.

The purpose of this paper is to construct, in ZFC, a new (tenth) subcontinuum of $\mathbb{H}^*$ and show how to use this continuum to raise, again in ZFC, the number of different subcontinua of $\mathbb{H}^*$ to fourteen. We do this by showing how, under CH, our continuum can be seen as a layer in a standard subcontinuum; because we have essentially four different layers in standard subcontinua (so far) this gives us four extra decomposable subcontinua. Under $\neg$CH we appeal to Dow [3] for at least six more decomposable subcontinua.

As a prelibation of things to come we give an outline of the rest of the paper. In Section 2 we show how, with the aid of certain subsets of ω^*, one can parameterize the indecomposable subcontinua of $\mathbb{H}^*$. In Section 3 we use this parameterization

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with a minimal closed and σ-invariant subset of ω^* as input to construct our new continuum. Here σ is the selfmap of ω^* determined by the shift on ω. We also show that our continuum has $2^{\mathfrak{c}}$ composants, thus providing a 'naturally occurring' continuum of this type. In Section 4 we prove in ZFC that $\mathbb{H}^*$ has at least fourteen topologically different subcontinua although the families differ according to whether CH holds or not. Finally, in Section 5 we make some remarks and pose some questions.

1. Preliminaries

In this section we establish some notation and recall some facts that will be needed later on; occasionally, as an aid to the reader, we give sketches of the arguments but we refer to Hart [7] for complete proofs and proper references. Additional basic topological material may be gleaned from Engelking's book [5]; Chapter Five of Kuratowski's book [8] is still one of the best references on continua — although it deals with metric continua, most of the general results are true *mutatis mutandis* for arbitrary continua.

1.1. **The shift on ω^*.** The shift σ on ω^* is the map determined by the usual shift on ω: $\sigma(n) = n+1$. For a point u of ω^* its image under σ is generated by the family $\{A+1 : A \in u\}$; because of this we sometimes write $u+1$ for $\sigma(u)$. Likewise we write $u-1$ for $\sigma^{\leftarrow}(u)$.

We will adopt the slightly nonstandard convention of calling a subset F of ω^* σ-invariant if $\sigma[F] \subseteq F$ *and* $\sigma^{\leftarrow}[F] \subseteq F$ (nonstandard because normally one takes σ-invariant to mean that $\sigma[F] \subseteq F$ only). The reason for this will become clear in Section 2; the closed and σ-invariant subsets occur naturally when we parameterize indecomposable subcontinua of $\mathbb{H}^*$.

We note that, with our interpretation, the set of accumulation points of a forward orbit is always σ-invariant. For let q be an accumulation point of $\operatorname{cl}\{p+n : n \in \omega\}$; there is an ultrafilter $u \in \omega^*$ such that $q = u\text{-}\lim(p+n)$. But then $q-1 = u\text{-}\lim\bigl(p+(n-1)\bigr)$ and $q+1 = u\text{-}\lim\bigl(p+(n+1)\bigr)$. Here, for a sequence $\langle x_n\rangle_n$ and $u \in \omega^*$, we say $x = u\text{-}\lim x_n$ iff for every neighbourhood U of x the set $\{n : x_n \in U\}$ belongs to u.

1.2. **The space $\mathbb{M}$.** The remainder $\mathbb{M}^*$ of the space $\mathbb{M} = \omega \times \mathbb{I}$, where $\mathbb{I}$ denotes the unit interval, determines the structure of $\mathbb{H}^*$ completely.

Before we show this we fix some notation. We abbreviate $\{n\}\times\mathbb{I}$ by $\mathbb{I}_n$ and we let $\pi : \beta\mathbb{M} \to \beta\omega$ be the map determined by $\pi(n,x) = n$. For $u \in \omega^*$ the fiber $\pi^{\leftarrow}(u)$ is denoted $\mathbb{I}_u$ — it is in a natural sense the u-th term of the sequence $\langle \mathbb{I}_n : n \in \omega\rangle$: in the hyperspace of closed subsets of $\beta\mathbb{M}$ one has $\mathbb{I}_u = u\text{-}\lim \mathbb{I}_n$. Similarly, if $x \in {}^\omega\mathbb{I}$ then we use x_u to denote the point $u\text{-}\lim\bigl\langle n, x(n)\bigr\rangle$ of $\mathbb{I}_u$.

The set $\mathbb{I}_u$ is a continuum that is irreducible between the two points $0_u = u\text{-}\lim\langle n, 0\rangle$ and $1_u = u\text{-}\lim\langle n, 1\rangle$, meaning that the only subcontinuum containing 0_u and 1_u is $\mathbb{I}_u$ itself. That $\mathbb{I}_u$ is irreducible between 0_u and 1_u follows from the fact that $\{x_u : x \in {}^\omega\mathbb{I}\} \setminus \{0_u, 1_u\}$ is a dense set of cut points of $\mathbb{I}_u$. There is a natural quasi-order on $\mathbb{I}_u$: one says $x \le_u y$ if every subcontinuum of $\mathbb{I}_u$ that contains 0_u and y also contains x. We shall employ the usual interval notation for this ordering. We note that if $x, y \in {}^\omega\mathbb{I}$ then $x_u \le_u y_u$ iff $\bigl\{n : x(n) \le y(n)\bigr\} \in u$.

This ordering $\le_u$ divides $\mathbb{I}_u$ into sets that we shall call layers; these are the equivalence classes under the relation

$$x \equiv y \qquad \text{iff} \qquad x \le_u y \text{ and } y \le_u x.$$

With a layer L one associates two subsets of ${}^\omega\mathbb{I}$:

$$A_L = \{a \in {}^\omega\mathbb{I} : L \cap \mathrm{cl}[0,a] = \emptyset\}$$

and

$$B_L = \{b \in {}^\omega\mathbb{I} : L \cap \mathrm{cl}[b,1] = \emptyset\}.$$

Here, for $x, y \in {}^\omega\mathbb{I}$, the symbol $[x,y]$ abbreviates $\bigcup_n \{n\} \times \bigl[x(n), y(n)\bigr]$.

The pair $\langle A_L, B_L\rangle$ represents a gap in the linearly ordered set ${}^\omega\mathbb{I}/u$, meaning that $a_u <_u b_u$ whenever $a \in A$ and $b \in B$. If there is $x \in {}^\omega\mathbb{I}$ that fills the gap (that is, $a_u <_u x_u <_u b_u$ for all $a \in A$ and $b \in B$) then L consists of the single point x_u.

The following fact will be used in 1.6 in the construction of several different continua in $\mathbb{H}^*$: if L is such that $[0_u, L)$ has countable cofinality then L consists of more than one point and the interval $(L, 1_u]$ has uncountable coinitiality.

One of the key points in the verification of this fact is that the closures of $[0_u, L)$ and $(L, 1_u]$ are $[0_u, L]$ and $[L, 1_u]$ respectively, so their intersection is L. Also, the interval $[0_u, L)$ has countable cofinality iff it is a cozeroset of $\mathbb{I}_u$. Because $\mathbb{I}_u$ is an F-space the sets $[0_u, L)$ and $(L, 1_u]$ cannot both be cozero sets. Finally, if $\langle a_n\rangle_n$ is a cofinal sequence in $[0_u, L)$ then its set of accumulation points is contained in L and, again because $\mathbb{I}_u$ is an F-space, homeomorphic to ω^*.

1.3. **Parameterizing** $\mathbb{H}^*$**.** The map $q : \mathbb{M} \to \mathbb{H}$ is defined by $q(n,x) = n + x$; this map is perfect and its Čech-Stone extension $q : \beta\mathbb{M} \to \beta\mathbb{H}$ maps $\mathbb{M}^*$ onto $\mathbb{H}^*$. We claim that q is one-to-one on every $\mathbb{I}_u$ and that the only identifications made by q on $\mathbb{M}^*$ are those of 1_u and 0_{u+1} for every $u \in \omega^*$.

One can deduce this from standard properties of the Čech-Stone compactification and the following two facts: 1) if $\bigl\{[a_n, b_n] : n \in \omega\bigr\}$ is a family of subintervals of $(0,1)$ then q maps the union $\bigcup_n \{n\} \times [a_n, b_n]$ homeomorphically onto the closed subset $\bigcup_n [n + a_n, n + b_n]$ of $\mathbb{H}^*$ and 2) the map q is exactly two-to-one on the set $\omega \times \{0,1\}$ (except at the point $\langle 0,0\rangle$).

One can define a map like q using any sequence $\boldsymbol{a}$ that increases to infinity: Simply put $q_{\boldsymbol{a}}(n,x) = a_n + x(a_{n+1} - a_n)$; we shall call $q_{\boldsymbol{a}}$ the parameterization of $\mathbb{H}^*$ determined by $\boldsymbol{a}$. The map q from the first paragraph will be called the *standard parameterization*.

1.4. **Standard subcontinua.** If $q_{\boldsymbol{a}} : \mathbb{M}^* \to \mathbb{H}^*$ is a parameterization then, as noted before, the map $q_{\boldsymbol{a}}$ is one-to-one on every continuum $\mathbb{I}_u$. Therefore $q_{\boldsymbol{a}}[\mathbb{I}_u]$ and $\mathbb{I}_u$ are homeomorphic; we call a subcontinuum of $\mathbb{H}^*$ of the form $q_{\boldsymbol{a}}[\mathbb{I}_u]$ a *standard subcontinuum* of $\mathbb{H}^*$.

The basic facts about standard subcontinua that we shall use here are that every proper subcontinuum of $\mathbb{H}^*$ is the intersection of the family of all standard subcontinua that contain it and that every nontrivial subcontinuum contains at least one standard subcontinuum (Theorem 2.6 substantially improves the last statement).

To prove the first statement consider a subcontinuum K and a point x not in it. First find neighbourhoods U of K and V of x with disjoint closures and then find a strictly increasing sequence $\boldsymbol{a}$ in $\mathbb{H}^*$ such that $U \cap \mathbb{H} \subseteq \bigcup_n [a_{2n}, a_{2n+1}]$ and $V \cap \mathbb{H} \subseteq \bigcup_n [a_{2n+1}, a_{2n+2}]$. One then readily checks that the family u of those subsets A of ω for which $K \subseteq \operatorname{cl} q_{\boldsymbol{a}}[\bigcup_{n\in A} \mathbb{I}_n]$ is an ultrafilter and that the standard subcontinuum $q_{\boldsymbol{a}}[\mathbb{I}_u]$ contains K but not x.

The second statement is proved in a similar fashion; one takes two distinct points x and y in the subcontinuum K and neighbourhoods U and V respectively with disjoint closures. It turns out that if we take $\boldsymbol{a}$ as above and find u such that $q_{\boldsymbol{a}}[\mathbb{I}_u]$ contains y by not x then in fact $q_{\boldsymbol{a}}[\mathbb{I}_u] \subseteq K$.

1.5. **Decomposable versus indecomposable.** A continuum is *decomposable* if it can be written as the union of two proper subcontinua and *indecomposable* otherwise. One can see whether a subcontinuum of $\mathbb{H}^*$ is decomposable or not by looking at its position inside standard subcontinua.

Lemma 1.1. *A subcontinuum of $\mathbb{H}^*$ is decomposable iff it is a nondegenerate interval of some standard subcontinuum.*

Here we take the intervals with respect to the quasi-order introduced before. It follows that decomposable subcontinua have cut points: intervals in standard subcontinua have (relative) interior and hence contain one of the cut points from Subsection 1.2. So, whenever an indecomposable continuum sits inside a standard subcontinuum it sits inside a layer of it. In fact layers of standard subcontinua are themselves indecomposable. Lemma 1.1 may also be used to prove the following one.

Lemma 1.2. *If of the two subcontinua K and L one is indecomposable and if their intersection is nonempty then either $K \subseteq L$ or $L \subseteq K$.*

We shall use Lemma 1.2 frequently in showing that one continuum must be contained in another one.

1.6. **The known subcontinua.** As noted in the introduction up to now nine nonhomeomorphic subcontinua of $\mathbb{H}^*$ were known in ZFC. Because we will want to show that the continuum that we will construct in Section 3 is new we give a brief description of the nine continua.

The first, K_1, consists of one point.

The next six will be constructed from a standard subcontinuum. Let us fix one such continuum $\mathbb{I}_u$. We take $K_2 = \mathbb{I}_u$. Next we take a layer L_1 such that the interval $[0_u, L_1)$ has countable cofinality and we let $K_3 = [0_u, L_1]$ and $K_4 = [L_1, 1_u]$. Next we take $L_2 <_u L_1$ such that $(L_2, 1_u]$ has countable coinitiality and we let $K_5 = [L_2, L_1]$. Finally we take $L_3 >_u L_1$ such that $(L_3, 1_u]$ has countable coinitiality as well and we let $K_6 = [L_2, L_3]$ and $K_7 = [L_1, L_3]$. These six continua are all decomposable and have two distinguished end sets; the number of one-point- or G_δ-end sets may be used to show that they are mutually nonhomeomorphic; for example K_3 and K_4 are nonhomeomorphic: K_3 has a one-point end set and a G_δ-end set, namely L_1, but in K_4 the end set L_1 is not a G_δ-set nor does it consist of one point.

The continuum K_8 will be L_1 and the ninth continuum K_9 is such that it has an increasing sequence $\langle C_n \rangle_n$ of proper indecomposable subcontinua such that $K_9 = \operatorname{cl} \bigcup_n C_n$. These two continua are indecomposable and distinguished by the presence of a proper dense F_σ-subset — K_9 has one and K_8 has none.

2. Making subcontinua from shift-invariant subsets of ω^*

In this section we describe a method by which all indecomposable subcontinua of $\mathbb{H}^*$ may be constructed. The idea is quite simple: For a subset F of ω^* we abbreviate $\bigcup_{u\in F}\mathbb{I}_u$ by $\mathbb{M}_F$ and we will denote the image of $\mathbb{M}_F$ under q by C_F. We shall identify F with the set $\{0_u : u\in F\}$ and also with the image of this set under q.

We shall see that C_F is an indecomposable continuum whenever F is closed, σ-invariant, dense-in-itself and not the union of two proper closed disjoint σ-invariant subsets. Conversely, if K is an indecomposable subcontinuum of $\mathbb{H}^*$ then there are a subset F of ω^* and a piecewise linear autohomeomorphism h of $\mathbb{H}$ such that $K = h[C_F]$.

The first lemma enables us to recognize σ-invariant subsets of ω^* by the behavior of $\mathbb{M}_F$ and C_F.

Lemma 2.1. *The set F is σ-invariant iff $\mathbb{M}_F = q^{\leftarrow}[C_F]$.*

Proof. Assume F is σ-invariant and let $x\in q^{\leftarrow}[C_F]$. Choose $y\in\mathbb{M}_F$ and $u\in F$ such that $q(y)=q(x)$ and $y\in\mathbb{I}_u$. If $y\neq x$ then either $y=0_u$ and $x=1_{u-1}$ or $y=1_u$ and $x=0_{u+1}$; in either case $x\in\mathbb{M}_F$ because $u-1,u+1\in F$.

Conversely assume $\mathbb{M}_F = q^{\leftarrow}[C_F]$ and let $u\in F$. Because $q(1_u)=q(0_{u+1})$ we must have $0_{u+1}\in\mathbb{M}_F$ but this means that $u+1\in F$. Likewise one concludes that $u-1\in F$. $\square$

Now we can prove the following proposition.

Proposition 2.2. *If C_F is an indecomposable continuum then F is closed and σ-invariant.*

Proof. Let $u\in F$. Because $q[\mathbb{I}_{u+1}]$ meets $q[\mathbb{I}_u]$ it also meets C_F. It cannot contain C_F because it does not contain $q[\mathbb{I}_u]$, therefore $q[\mathbb{I}_{u+1}]\subseteq C_F$ and so $u+1\in F$. Likewise one shows that $u-1\in F$.

Finally, F is closed because $F=\pi\bigl[q^{\leftarrow}[C_F]\bigr]$ and π is a closed map. $\square$

For convenience we shall henceforth assume that F is a closed and σ-invariant subset of ω^*. We determine when the space C_F is connected. In the proof of the next theorem $A=B\oplus C$ denotes that A is the disjoint union of the non-empty closed subsets B and C.

Theorem 2.3. *The set C_F is connected iff F cannot be written as the union of two disjoint proper closed σ-invariant subsets.*

Proof. Necessity: If $F=G\oplus H$ with G and H both σ-invariant then $\mathbb{M}_F = \mathbb{M}_G\oplus\mathbb{M}_H$, hence $C_F=C_G\oplus C_H$ by Lemma 2.1.

Sufficiency: If $C_F=G\oplus H$ then we can divide F into two sets $G'=\{u : q[\mathbb{I}_u]\subseteq G\}$ and $H'=\{u : q[\mathbb{I}_u]\subseteq H\}$. Because every $\mathbb{I}_u$ is connected this is a partition of F.

For every $u\in F$ the set $q[\mathbb{I}_{u-1}]\cup q[\mathbb{I}_u]\cup q[\mathbb{I}_{u+1}]$ is connected, so if u is in G' (or H') then so are $u-1$ and $u+1$. This shows that G' and H' are σ-invariant.

To see that G' and H' are closed just observe that $G=C_{G'}$ and so (again by the lemma) $G'=\pi\bigl[q^{\leftarrow}[G]\bigr]$ and likewise for H. $\square$

Next we determine when for a set F as in Theorem 2.3 the continuum C_F is an indecomposable continuum.

Lemma 2.4. *If $u \in F$ then u is an isolated point of F iff $q[\mathbb{I}_u]$ has nonempty interior in C_F.*

Proof. If u is isolated in F then $O = \mathbb{I}_u \setminus \{0_u, 1_u\}$ is open in $\mathbb{M}_F$ and it satisfies $O = q^{\leftarrow}\bigl[q[O]\bigr]$, so its image is open in C_F and contained in $q[\mathbb{I}_u]$.

Conversely, if there is an open set O in $\beta\mathbb{M}$ such that $\emptyset \neq O \cap C_F \subseteq q[\mathbb{I}_u]$ then $q^{\leftarrow}[O]$ does the same thing for $\mathbb{I}_u$ with respect to $\mathbb{M}_F$. One can then find $a, b \in {}^{\omega}\mathbb{I}$ such that $a_u <_u b_u$ and $[a_u, b_u] \subseteq O$. By compactness there is $U \in u$ such that the closure of

$$\bigcup_{n \in U} \{n\} \times \bigl[a(n), b(n)\bigr]$$

is contained in O. This then implies that $U \cap F = \{u\}$. □

Now we can characterize indecomposability of C_F.

Theorem 2.5. *If C_F is a continuum then it is indecomposable iff F is dense-in-itself.*

Proof. We just proved necessity: If $u \in F$ is isolated then $\mathbb{I}_u$ is a subcontinuum of C_F with nonempty interior (if $F = \{u\}$ then certainly $C_F = \mathbb{I}_u$ is decomposable).

To prove sufficiency assume that C_F is decomposable. By Lemma 1.1 we can find a standard subcontinuum $\mathbb{J}_u$ such that C_F has nonempty interior in $\mathbb{J}_u$ and therefore contains a cut point x of $\mathbb{J}_u$. Choose $v \in F$ such that $\mathbb{I}_v$ contains x. Then $\mathbb{I}_v$ is a subinterval of $\mathbb{J}_u$ and hence has nonempty interior in $\mathbb{J}_u$. So certainly $\mathbb{I}_v$ has nonempty interior in C_F; it follows that v is an isolated point of F. □

We conclude that we get an indecomposable continuum in $\mathbb{H}^*$ each time we get a dense-in-itself σ-invariant subset F of ω^* as in Theorem 2.3. The converse holds as well.

Theorem 2.6. *Let K be a nontrivial indecomposable subcontinuum of $\mathbb{H}^*$. Then there are a σ-invariant subset F of ω^* and a (piecewise linear) autohomeomorphism h of $\mathbb{H}$ such that $K = \beta h[C_F]$.*

Proof. Take a standard subcontinuum that is contained in K, that is, take a parameterization $q_{\boldsymbol{a}}$ of $\mathbb{H}$ and $u \in \omega^*$ such that $q_{\boldsymbol{a}}[\mathbb{I}_u] \subseteq K$.

We claim that for every point $v \in \omega^*$ either $q_{\boldsymbol{a}}[\mathbb{I}_v] \subseteq K$ or $q_{\boldsymbol{a}}[\mathbb{I}_v] \cap K = \emptyset$. For if $q_{\boldsymbol{a}}[\mathbb{I}_v]$ meets K then it cannot contain K as it cannot contain $q_{\boldsymbol{a}}[\mathbb{I}_u]$, so it must be contained in K.

Now we consider $F = \{v \in \omega^* : q_{\boldsymbol{a}}[\mathbb{I}_v] \subseteq K\}$. From what we have established above it follows that $K = q_{\boldsymbol{a}}[\mathbb{M}_F] = \beta h[C_F]$, where $h : \mathbb{H} \to \mathbb{H}$ is the unique piecewise linear map for which $h \circ q = q_{\boldsymbol{a}}$.

Because C_F apparently is an indecomposable continuum we see that F must be σ-invariant. □

Remark 2.7. Let us note that we cannot distinguish the nontrivial subcontinua of $\mathbb{H}^*$ by means of cellularity, density or (π-)weight.

As $\mathbb{H}^*$ has weight $\mathfrak{c}$ it suffices to show that each nontrivial subcontinuum has cellularity $\mathfrak{c}$. Indeed, for every $x \in (0,1)$ we can take the open subset O_x of $\mathbb{M}^*$ determined by $\bigcup_n \{n\} \times (x - 2^{-n}, x + 2^{-n})$; the family $\bigl\{O_x : x \in (0,1)\bigr\}$ is pairwise disjoint and every O_x meets every $\mathbb{I}_u$.

Now if K is a subcontinuum of $\mathbb{H}^*$ then we take a parameterization $q_{\boldsymbol{a}} : \mathbb{M} \to \mathbb{H}$ and $u \in \omega^*$ such that $q_{\boldsymbol{a}}[\mathbb{I}_u] \subseteq K$. Then $\bigl\{q[O_x] \cap K : x \in (0,1)\bigr\}$ is a family of pairwise disjoint nonempty open subsets of K. (Note that $O_x = q_{\boldsymbol{a}}^{\leftarrow}\bigl[q_{\boldsymbol{a}}[O_x]\bigr]$ for every x, so that $q_{\boldsymbol{a}}[O_x]$ is open in $\mathbb{H}^*$.)

3. A new continuum

In this section we apply the construction from Section 2, with very special input, to construct a new subcontinuum of $\mathbb{H}^*$. The input that we shall use is a minimal closed σ-invariant subset of ω^*. Such sets are easily constructed: Apply Zorn's Lemma to the family of all closed σ-invariant subsets of ω^*.

It is readily verified that a closed set F is minimally σ-invariant iff for every $u \in F$ the orbit $O_u = \{u + n : n \in \mathbb{Z}\}$ is dense in F. From this one easily deduces that for every $u \in F$ the set $S_u = \bigcup_n q[\mathbb{I}_{u+n}]$ is proper dense F_σ-subset of C_F.

Each minimal σ-invariant subset is clearly indecomposable and dense-in-itself so it follows that C_F is an indecomposable continuum whenever F is closed and minimally σ-invariant. The following theorem shows that C_F is indeed new.

Theorem 3.1. *If F is a minimal closed σ-invariant subset of ω^* then C_F is not homeomorphic to any one of the continua K_1 through K_9.*

Proof. Since K_1 consists of one point and since the continua $K_2, \dots, K_7$ are decomposable we only have to deal with K_8 and K_9. Now K_8 has the property that every nonempty G_δ-subset of it has nonempty interior but K_9 and C_F do not have this property, so we are left with K_9.

Remember that K_9 has a proper, dense and meager F_σ-subset $H = \bigcup_n C_n$, where each C_n is indecomposable and nowhere dense in C_{n+1}. Assume $h : K_9 \to C_F$ is a homeomorphism and choose $x \in h[C_0]$; also pick $u \in F$ such that $x \in q[\mathbb{I}_u]$.

Because $h[H]$ is dense in C_F we know that there is an n such that $h[C_n] \nsubseteq q[\mathbb{I}_u]$ and consequently — because C_n is indecomposable — $q[\mathbb{I}_u] \subseteq h[C_n]$. But now one shows inductively that $q[\mathbb{I}_{u+k}] \subseteq h[C_n]$ for every k and hence that $S_u \subseteq h[C_n]$.

This leads to a contradiction since S_u is dense in C_F and C_n is nowhere dense in K_9. □

Henceforth we shall denote the continuum of Theorem 3.1 by K_{10}.

The *composant* of a point x in a continuum X is the union of all proper subcontinua of X that contain the point. In an indecomposable continuum the composants are the equivalence classes under the relation "there is a proper subcontinuum containing both x and y".

The following proposition identifies the composants of K_{10}. In its proof we apply Theorem 5.3 from Hart [7] (due to van Douwen) which gives exact information on how two standard subcontinua intersect (if they intersect). Indeed if K and L are standard subcontinua, K with end points a and b, and L with end points c and d that intersect then $K \cap L$ equals L if $c, d \in K$, it equals K if $c, d \notin L$, and it equals one of the standard subcontinua $[a, d]$, $[c, b]$, $[a, c]$ and $[d, b]$ depending on which of the end points of L is in K and which of the end points of K is in L. Also, if K and L intersect then $K \cup L$ is a standard subcontinuum as well.

Proposition 3.2. *If $u \in F$ then S_u is the composant of u in K_{10}.*

Proof. Denote the composant of u by K_u. Clearly $S_u \subseteq K_u$, because S_u is the union of a family of proper subcontinua of K_{10} that contain the point u.

Conversely let K be a nontrivial proper subcontinuum of K_{10} that contains u. If K is indecomposable then either $K \subseteq q[\mathbb{I}_u]$ or $q[\mathbb{I}_u] \subseteq K$. The latter case will give rise to a contradiction; for much as in the proof of Proposition 2.2 we can show that for each $n \in \mathbb{Z}$ if $q[\mathbb{I}_{u+n}] \subseteq K$ then also $q[\mathbb{I}_{u+n-1}] \subseteq K$ and $q[\mathbb{I}_{u+n+1}] \subseteq K$. This would imply that $K_{10} = \operatorname{cl} S_u \subseteq K$.

So assume K is decomposable and choose a standard subcontinuum $\mathbb{J}_v$ such that K is a nondegenerate interval of it. Now if for all $n > 0$ we would have $u + n \in \mathbb{J}_v$ then, by the remarks preceding this proposition we would also have $q[\mathbb{I}_{u+n}] \subseteq \mathbb{J}_v$ for all $n \in \omega$. But then

$$K_{10} \subseteq \operatorname{cl} \bigcup_{n \geq 0} q[\mathbb{I}_{u+n}] \subseteq \mathbb{J}_v.$$

Now $K \subseteq K_{10}$ implies that K_{10} would be a nondegenerate interval of $\mathbb{J}_v$ and hence decomposable.

We conclude that there is $n > 0$ such that $u+n \notin \mathbb{J}_v$ and likewise that $u+m \notin \mathbb{J}_v$ for some $m < 0$. Applying the remarks preceding the proposition again we see that $\mathbb{J}_v$ is contained in the standard subcontinuum $\bigcup_{m \leq i < n} q[\mathbb{I}_{u+i}]$ and hence that $K \subseteq S_u$. $\square$

Thus composants of K_{10} correspond to orbits in F and we see that K_{10} has $2^{\mathfrak{c}}$ composants — the first continuum with $2^{\mathfrak{c}}$ composants was constructed by Smith in [9]. This is of interest because the number of composants of $\mathbb{H}^*$ and of all countable cofinality layers is independent of ZFC: Under CH the number is $2^{\mathfrak{c}}$ and under NCF the number is 1 — we refer to Hart [7] for references. We do not know the number of composants of K_9.

4. Four more subcontinua from CH

In this section we show how our construction may be employed to produce, using CH, four more subcontinua of $\mathbb{H}^*$. At the end of this section we use a result of Dow to show that under $\neg\mathsf{CH}$ there are always at least six continua different from K_1 through K_{10}.

We shall show that, under CH, the continuum K_{10} can be constructed as a layer in some standard subcontinuum $\mathbb{I}_u$. To this end we first investigate how we can make the parameterization of the layers as in Section 2 more concrete.

4.1. **Concrete descriptions of layers.** We fix a nontrivial layer L of some standard subcontinuum $\mathbb{I}_u$ and we find F such that $L = C_F$ (more precisely, we find a parameterization $q_{\boldsymbol{a}}$ of $\mathbb{H}$ and $F \subseteq \omega^*$ such that $L = q_{\boldsymbol{a}}[\mathbb{M}_F]$).

Consider the gap $\langle A_L, B_L\rangle$ determined by L. Now, because L is nontrivial we can find a function $f : \omega \to \mathbb{N}$ such that $\{n : b(n) - a(n) > 2^{-f(n)}\}$ belongs to u whenever $a \in A_L$ and $b \in B_L$.

Instead of ω we consider the set $K_f = \{\langle n, m\rangle : m < 2^{f(n)}\}$, and order it lexicographically. The shift σ on K_f is defined by

$$\sigma(n, m) = \begin{cases} \langle n, m+1\rangle & \text{if } m+1 < 2^{f(n)} \text{ and} \\ \langle n+1, 0\rangle & \text{otherwise.} \end{cases}$$

Furthermore put, for $m < f(n)$,

$$\mathbb{I}_{n,m} = \{n\} \times \left[m \cdot 2^{-f(n)}, (m+1) \cdot 2^{-f(n)}\right].$$

We define a filter $\mathcal{F}_L$ on K_f as follows:

$$\mathcal{F}_L = \{P : L \subseteq \mathrm{cl}_{\beta\mathbb{M}} \bigcup_{\langle n,m\rangle\in P} \mathbb{I}_{n,m}\}.$$

This filter has a base consisting of sets of the form

$$[g,h,U] = \bigl\{\langle n,m\rangle : g(n) \le m \le h(n), n \in U\bigr\},$$

where g and h are such that $g\cdot 2^{-f} \in A_L$, $h\cdot 2^{-f} \in B_L$ and $U \in u$. From this it follows that for every subset P of K_f

$$P \in \mathcal{F}_L \qquad \text{iff} \qquad \sigma[P] \in \mathcal{F}_L.$$

We can now describe F; it is the set $\bigcap_{P\in\mathcal{F}_L} P^*$ in K_f^*. Then $L = q^*[\mathbb{M}_F]$.

An alternative description of F is as follows. First we consider the map $\pi : K_f \to \omega$ defined by $\pi(n,m) = n$ and its Čech-Stone extension $\pi : \beta K_f \to \beta\omega$. The set $T_u = \pi^{\leftarrow}(u)$ contains a dense set of isolated points: Those determined by graphs of functions below f. For if $g \in {}^{\omega}\omega$ is such that $g(n) \le 2^{f(n)}$ for all n then the closure of its graph meets T_u in exactly one point and this point is therefore isolated. This set of isolated points is linearly ordered by $\le_u$ and the gaps in this order are filled — in K_f^* — by sets like F. One might call F a layer in the space T_u.

4.2. **A new layer.** We shall now construct, assuming CH, an ultrafilter u, a function f and a layer F in T_u such that F is minimally σ-invariant. The corresponding continuum C_F is then a layer in $\mathbb{I}_u$

The following lemma provides a way to recognize minimal σ-invariant sets.

Lemma 4.1. *A closed and σ-invariant set F is minimal iff for every open set that meets F there are finitely many translates that cover F.*

Proof. If O is open and meets the minimal set F then the family $\bigl\{\sigma^n[O] : n \in \mathbb{Z}\bigr\}$ covers F, because every orbit is dense and meets O. As F is compact this family has a finite subcover.

Conversely, if F is not minimal and G is a proper closed σ-invariant subset of F then $\omega^*\setminus G$ is an open set that meets F and is equal to all of its translates. □

Using this lemma we can describe what we need to construct. We must find an ultrafilter u and a gap $\langle\mathcal{F},\mathcal{G}\rangle$ in ${}^{\omega}\omega/u$ below some function h, with the following property: if $X \subseteq K_h$ meets every set of the form $[f,g,U]$ with $f \in \mathcal{F}$, $g \in \mathcal{G}$, $U \in u$ then there are $f \in \mathcal{F}$, $g \in \mathcal{G}$, $U \in u$ and $k \in \omega$ such that $[f,g,U] \subseteq \bigcup_{|i|\le k} \sigma^i[X]$. Here, as above

$$[f,g,U] = \bigl\{\langle n,m\rangle : f(n) \le m \le g(n), n \in U\bigr\}$$

for $f,g \in {}^{\omega}\omega$ and $U \subseteq \omega$.

The point is that the family of sets $[f,g,U]^*$ is a neighbourhood base for the set F, so that the previous paragraph expresses the condition of Lemma 4.1 in terms of the natural base for K_h^*.

The desired gap may be constructed, using CH, in an induction of length ω_1. One constructs $\mathcal{F} = \{f_\alpha : \alpha < \omega_1\}$, $\mathcal{G} = \{g_\alpha : \alpha < \omega_1\}$ and $\{U_\alpha : \alpha < \omega_1\}$ such that

1. $\{U_\alpha : \alpha < \omega_1\}$ is decreasing mod finite and generates a P-point u.
2. The pair $\langle\mathcal{F},\mathcal{G}\rangle$ is a gap below the identity function Id in ${}^{\omega}\omega/u$.

To take care of our condition and to ensure that we get a gap we enumerate the family of subsets of K_h as $\{X_\alpha : \alpha < \omega_1\}$ and ${}^\omega\omega$ as $\{h_\alpha : \alpha < \omega_1\}$. We give a rough sketch of the construction.

We start the induction by letting f_0 be identically 0, $g_0 = \mathrm{Id}$ and $U_0 = \omega$. We make sure that for every α one has

(1) $\sup\bigl\{g_\alpha(n) - f_\alpha(n) : n \in U_\alpha\bigr\} = \omega$ (to keep the induction alive).
(2) For all $n \in U_{\alpha+1}$ either $h_\alpha(n) < f_{\alpha+1}(n)$ or $h_\alpha(n) > g_{\alpha+1}(n)$ (to create a gap).
(3) If no finite number of translates of X_α covers $[f_\alpha, g_\alpha, U_\alpha]$ then X_α is disjoint from $[f_{\alpha+1}, g_{\alpha+1}, U_{\alpha+1}]$ (to meet our condition for minimality).

All steps are routine except when we have to take care of (3).

Assume no finite number of translates of X_α covers the set $[f_\alpha, g_\alpha, U_\alpha]$ and consider for each $n \in U_\alpha$ the longest interval I_n in $\bigl\{\langle n, m\rangle : m \le h(n)\bigr\}$ that is contained in $[f_\alpha, g_\alpha, U_\alpha] \setminus X_\alpha$. The claim is that $\sup_{n\in U_\alpha} |I_n| = \omega$. For if the supremum equals $k < \omega$ then $\bigcup_{|i|\le k} \sigma^i[X_\alpha]$ contains $[f_\alpha, g_\alpha, U_\alpha]$. This enables us to find functions p and q between f_α and g_α such that $[p, q, U_\alpha] \cap X_\alpha = \emptyset$.

Further modifications of p, q and U_α will be necessary so as to satisfy (2) and to ensure that the U_β will in the end generate an ultrafilter.

4.3. **The four extra continua.** It is clear that we can construct infinitely many K_{10}-like layers in $\mathbb{I}_u$ just by varying the gap. This implies that we can take four layers L_1, L_2, L_3 and L_4 in $\mathbb{I}_u$ such that $[0_u, L_1)$ and $[0_u, L_3)$ have countable cofinality, L_2 and L_4 are like K_{10} and $L_1 <_u L_2 <_u L_3 <_u L_4$.

We claim that the none of the four intervals $[0_u, L_2]$, $[L_1, L_2]$, $[L_2, L_3]$ and $[L_2, L_4]$ is homeomorphic to one of the continua K_1 through K_{10}.

As the intervals are nontrivial and decomposable they are different from K_1, K_8, K_9 and K_{10}. They are also different from K_2 through K_7: none of these have an end set with a proper dense F_σ-subset.

Finally, that the intervals are mutually nonhomeomorphic can be seen by looking at the 'other' end set: one point (0_u), not a G_δ-set (L_1), a G_δ-set (L_3) and not a G_δ-set but not homeomorphic to L_1 (L_4).

4.4. **Six extra continua from $\neg\mathsf{CH}$.** In [3] Dow proved a general result on ultrapowers that, when interpreted in terms of the continua $\mathbb{I}_u$, produces for every regular uncountable cardinal $\kappa \le \mathfrak{c}$ an ultrafilter u_κ such that

- if L is a layer in $\mathbb{I}_{u_\kappa}$ such that $[0_{u_\kappa}, L)$ has countable cofinality then $(L, 1_{u_\kappa}]$ has coinitiality κ.

We let $u = u_{\omega_1}$ and $v = u_{\omega_2}$, and use $\mathbb{I}_u$ to make K_2 through K_7 and $\mathbb{I}_v$ to make the corresponding K_2' through K_7'. Then the K_i are mutually nonhomeomorphic, the K_i' are mutually nonhomeomorphic, and no K_i is homeomorphic to any K_j' because K_i has layers of type $\langle\omega, \omega_1\rangle$ whereas K_j' has none.

5. Concluding remarks

We have, in ZFC, ten 'honest' different subcontinua of $\mathbb{H}^*$. Under CH we can produce four more and under $\neg\mathsf{CH}$ even six more. Thus, ZFC proves that $\mathbb{H}^*$ has at least fourteen different subcontinua.

This should not be the best we can get. One would expect, given the way things usually go with the Čech-Stone compactification, to find $2^{\mathfrak{c}}$ different subcontinua in $\mathbb{H}^*$ but at present this number seems very hard to reach.

For instance, in Dow and Hart [4] the authors have shown that under CH all standard subcontinua of $\mathbb{H}^*$ are homeomorphic, as are all layers that determine an interval with countable cofinality.

Although trying to vary minimal σ-invariant subsets of ω^* may look promising at first, one should realize that in [1] Balcar and Błaszczyk showed that all such sets are all homeomorphic to the absolute of the Cantor cube of weight $\mathfrak{c}$

This leads to the (depressing) conjecture that under CH all versions of K_{10} are homeomorphic.

A positive answer to the following question would give us five more intervals.

Question 5.1. Can the K_9-type continua be (homeomorphic to) layers of standard subcontinua?

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