What is ‘Finite’?

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When you search the website of the Dutch Science Agenda (`https://vragen.wetenschapsagenda.nl/`) for the keywords ‘oneindig’ and ‘oneindigheid’ (infinite and infinity) you will get more hits than for ‘eindig’ and ‘eindigheid’(finite and finiteness): 25 against 6. Apparently people find the notion of the infinite more difficult, interesting, fascinating, ..., than that of the finite. Mathematically the difference need not be so large: ‘infinite’ is ‘not finite’ and ‘finite’ is ‘not infinite’, so if you know one you know the other. The question then is: “What is ‘finite’?” Once we have an answer to that question we also know what ‘infinite’ is: just put ‘not’ in front of that answer. Unfortunately (or maybe fortunately) it is not quite that simple; it will turn out that what we want from both notions makes it difficult, at first, to make them simply each others negations.

The dictionary

It is always interesting to look in a dictionary for the ‘normal’ meaning of a mathematical term; often that will also give an idea why students find it hard to get to grips with a concept: that ‘normal’ meaning quite often has no bearing on what mathematics wants to say.
I use *The Chambers Dictionary* ([1]) when doing crosswords, so I looked there for definitions of ‘finite’ and ‘infinite’.

finite *adj* having an end or limit; subject to limitations or conditions, opp to *infinite*. [l. *finitus*, pap of *finire* to limit]

infinite *adj* without end or limit; greater than any quantity that can be assigned [*maths*]; extending to infinity; vast; in vast numbers; inexhaustible; infinitated (*logic*)

infinitate *vt* to make infinite; to turn into a negative term (*logic*).

So, the first definitions of ‘finite’ and ‘infinite’ are each other’s negations but then there is a slight divergence. ‘Finite’ is the opposite of ‘infinite’, but not vice versa; also, ‘infinite’ has more variety than finite. I included the verb ‘infinitate’ because I had not seen it before and because it has a nice logical constant as well.
There is one noteworthy bit among the definitions of ‘infinite’: the phrase “greater than any quantity that can be assigned” comes more or less straight out of Euclid’s *Elements*:

Book IX, Proposition 20. The primes are more than any assigned multitude of prime numbers.

Nowadays we formulate this as: “There are infinitely many prime numbers.”

Mathematics

The problem with dictionary definitions is that they use words that have their own definition and quite often looking up those definitions will lead you to others and, ..., after a while you find yourself going round in circles. The definition of ‘infinite’ that goes back to Euclid contains ‘greater’, ‘quantity’, and ‘assigned’. We need to give meaning to these when we want to make the definition unambiguous.
What we will do is define ‘finite’ in such a way that ‘infinite’, as its negation, will let Euclid’s formulation make sense.
Before we do that first an aside: in Analysis you sometimes read about ‘finite intervals’; those are simply intervals with real numbers as limits. For example $[0,1]$ and $(2,10^{220}]$ are finite intervals, but not $(0,\to)$. This corresponds to the dictionary definition of ‘finite’ but we will not discuss this type of finiteness because it does not occur that often and because these intervals do have an infinite number of elements.

Finite sets

So what is the definition of ‘finite set’? A bit anticlimactic maybe, but a set X is *finite* if there are a natural number n and a bijection $f : n \to X$. Before anyone panics, “a bijection between a set and natural number?”: in Set Theory we define $\mathbb{N}$ in such a way that $n = \{i \in \mathbb{N} : i < n\}$. This is because we want to keep things a simple as possible and this is indeed the simplest possible way to define natural numbers *set-theoretically*, see [5]. So, for example: $0 = \emptyset$, $1 = \{0\}$, $2 = \{0,1\}$, ..., $7 = \{0,1,2,3,4,5,6\}$, ...; in general: $n+1 = n \cup \{n\}$.
It is a good exercise to show that there is at most one such n.

Exercise. Prove, for every n, that there is no bijection between n and $n+1$.

The consequence is now that we can *define* for finite sets what the *number of elements* is: the unique n for which the desired bijection exists.

Infinite sets

Now we also know what an infinite set is: one for which no n as in the definition of ‘finite set’ can be found. That is quite negative; in fact you are basically empty-handed when your set is infinite: you have no n and no bijection.
What you do have is lots of injections: if a set X is infinite then you can prove, by induction, that for every $n \in \mathbb{N}$ there is an injection from n into X. This may seem completely self-evident — “just pick some points” — but it takes a bit of work to turn those words into a proper proof, and we will see later why this might need some work.
For now we note that Euclid’s proof of his theorem, that you can look up on-line, see [3], shows indeed that the set P of prime numbers is infinite according to our definition: it shows that for every n no injection $f : n \to P$ is surjective.

Alfred Tarski

In [6] the Polish mathematician Alfred Tarski made a thorough investigation of the notion of a finite set. One thing he asked himself was whether you could definite finiteness without referring directly to natural numbers.

Figure 1: Alfred Tarski

The answer was "yes, that is possible". Nowadays we call a set X *Tarski-finite* is the following holds: every nonempty family $\mathcal{A}$ of subsets of X has a *maximal element*. Note: 'maximal' means that there is no larger element (and that is not the same as being the largest element). For example in the family $\{\{x\} : x \in X\}$ *every* member is maximal, but there is no maximum (if X has more than one element).

Exercise. Prove, by induction, that every natural number is Tarski-finite (and hence that ever finite set is Tarski-finite).

Once you have done this exercise you should, of course, do the following one too.

Exercise. Prove that every Tarski-finite set is finite.

Here is a hint: let $\mathcal{A}$ be the family of all finite subsets of X. It is non-empty because $\emptyset \in \mathcal{A}$. Next prove that a maximal element of $\mathcal{A}$ must be a *maximum* and indeed that it must be X itself.
This characterization of finiteness, which we might also call *The Maximality Principle*, is very useful in mathematics, especially in Combinatorics.

Richard Dedekind

One of the first mathematicians who sought to define finite and infinite sets was Richard Dedekind. His most famous definition is from [2]. It even made it into the dictionary:

infinite set *n* (*maths*) a set that can be put into one-one correspondence with part of itself

In terms of maps: a set X is *Dedekind-infinite* is there is an injective map $f : X \to X$ that is *not* surjective. This property has two nice equivalences, especially number 2 is something that we would like to be true about infinite sets.

Exercise. Prove that the following three statements are equivalent.

1. X is Dedekind-infinite
2. there is an injective map $f : \mathbb{N} \to X$
3. there is a bijection $f : X \to X \cup \{p\}$ for (some) p not in X

Now, by definition, a set X is *Dedekind-finite* if every injective map $f : X \to X$ is surjective.

Exercise. Prove, by induction, that every natural number is Dedekind-finite (and hence that every finite set is Dedekind-finite).

Figure 2: Richard Dedekind

We now have a problem; our notion of 'finite' is very practical: a finite set comes with an enumeration $f : n \to X$ and we can use that in our proofs. Similarly a Dedekind-infinite set comes with an injective map $f : \mathbb{N} \to X$, also quite handy to have.

Unfortunately, '(in)finite' is not the same as 'Dedekind-(in)finite'.

It seems easy to prove that every infinite set is Dedekind-infinite: pick a point x_0, the set is infinite so there is another point x_1, there is another point x_2, …, and so on. The map that send n to x_n is injective.

Well, …, no; there is no way to turn the "…, and so on" into a formal set-theoretic proof that involves only finitely many steps. By 'formal proof' I mean a proof as described in the course Logic (TW3520) that you can take every year (recommended).

To give a proof in finitely many steps you need an extra assumption, the Axiom of Choice ([7]).

Cyclic permutations

In the foreword to the second edition of [2] Dedekind mentioned another definition of 'finite': a set X is Dedekind*-finite if there is a map $f : X \to X$ such that if $A \subseteq X$ is such that $f[A] \subseteq A$ then $A = \emptyset$ or $A = X$.

It is not hard to show that natural numbers are Dedekind*-finite: think of cyclic permutations.

On the other hand: if X is Dedekind*-finite then X is in fact finite, and the map f is indeed a cyclic permutation.

Exercise. Prove this. *Hint*: take some $x \in X$ and prove that $x = f^n(x)$ for some $n \geq 1$.

So, Dedekind had two definitions of 'finite' (and of 'infinite') and he tried quite hard to prove that they were equivalent but did not succeed. If you want to know why then you should study [4].

References

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[4] Thomas J. Jech, *The axiom of choice*, Studies in Logic and the Foundations of Mathematics, Vol. 75. (1973) North-Holland Publishing Co., Amsterdam.

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