



OUR GOAL: PROVE THAT $\dim [0,1]^n = n$
AND $\dim \mathbb{R}^n = n$.

FIRST: $\dim [0,1]^n = \dim \mathbb{R}^n$

THIS MEANS WE CAN THEN CONCENTRATE ON $[0,1]^n$
WHICH IS COMPACT.

LEMMA: IF F IS A CLOSED SUBSPACE OF X
THEN $\dim F \leq \dim X$. (SO $\dim [0,1]^n \leq \dim \mathbb{R}^n$)

PROOF WE SHOW: IF $\dim X \leq n$ THEN $\dim F \leq n$.

(THIS IS HOW MANY PROOFS IN DIMENSION THEORY
WORK.)

LET $\{(A_0, B_0), \dots, (A_n, B_n)\}$ BE A FAMILY OF
 $n+1$ PAIRS OF DISJOINT CLOSED SETS IN F .
THEN $\{(A_0, B_0), \dots, (A_n, B_n)\}$ IS ALSO SUCH A
FAMILY IN X .

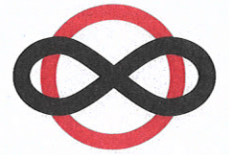
IN X THERE ARE PARTITIONS P_0, \dots, P_n
(P_i BETWEEN A_i AND B_i) SUCH THAT
 $P_0 \cap P_1 \cap \dots \cap P_n = \emptyset$

SO $X \setminus P_i = U_i \cup V_i$ WITH U_i, V_i OPEN AND
DISJOINT AND $A_i \in U_i$ AND $B_i \in V_i$.

INTERSECT EVERY THING WITH F :

$Q_i = P_i \cap F$ IS A PARTITION IN F BETWEEN
 A_i AND B_i : $A_i \in U_i \cap F$, $B_i \in V_i \cap F$

AND, OF COURSE, $Q_0 \cap \dots \cap Q_n = \emptyset$.



To prove $\dim \mathbb{R}^n \leq \dim [0, 1]^n$ REQUIRES A BIT MORE WORK.

LEMMA IF $\{F_0, F_1, \dots, F_n\}$ IS A FAMILY OF CLOSED SETS WITH $F_0 \cap F_1 \cap \dots \cap F_n = \emptyset$ THEN THERE ARE OPEN SETS O_0, O_1, \dots, O_n WITH $\bar{O}_0 \cap \bar{O}_1 \cap \dots \cap \bar{O}_n = \emptyset$.

PROOF BY INDUCTION

$$n=1 \quad F_0 \cap F_1 = \emptyset$$

USE THE METRIC AND DEFINE

$$f: X \rightarrow \mathbb{R} \text{ BY } f(x) = \frac{d(x, F_0)}{d(x, F_0) + d(x, F_1)}$$

THEN:

- f IS CONTINUOUS (EXERCISE)

- IF $x \in F_0$ THEN $f(x) = 0$

- IF $x \in F_1$ THEN $f(x) = 1$

$$\text{LET } O_0 = f^{-1}([0, 1/2]) \text{ AND } O_1 = f^{-1}([2/3, 1])$$

$$\text{THEN } \bar{O}_0 \cap \bar{O}_1 \subseteq f^{-1}([0, 1/2]) \cap f^{-1}([2/3, 1]) = \emptyset.$$

$n \rightarrow n+1$ LET F_0, \dots, F_n, F_{n+1} BE GIVEN

APPLY THE CASE $n=1$ TO

$$G = F_0 \cap \dots \cap F_n \text{ AND } F_{n+1}$$

LET U AND V BE OPEN WITH

$$G \subseteq U, F_{n+1} \subseteq V \text{ AND } \bar{U} \cap \bar{V} = \emptyset.$$

APPLY THE INDUCTIVE HYPOTHESIS TO

$$F_0 \setminus U, \dots, F_n \setminus U$$

TO GET OPEN SETS O_0, \dots, O_n WITH

$$F_i \setminus U \subseteq O_i \text{ (ALL } i) \text{ AND } O_0 \cap \dots \cap O_n = \emptyset$$

$$\text{LET } U_i = O_i \cup \emptyset \quad i \leq n$$

$$U_{n+1} = V$$

ASSUME $x \in \bar{U}_0 \cap \dots \cap \bar{U}_n \cap \bar{U}_{n+1}$

THEN $x \notin \bar{O}_0 \cap \dots \cap \bar{O}_n$, SO $x \in \bar{U}$

BUT THEN $x \notin \bar{V}$

$$\text{WE FIND } \bar{U}_0 \cap \dots \cap \bar{U}_n \cap \bar{U}_{n+1} = \emptyset.$$



THE COUNTABLE-CLOSED-SUM THEOREM

ASSUME X IS THE UNION OF COUNTABLY MANY CLOSED SETS $\{F_\alpha : \alpha \in \mathbb{N}\}$ SUCH THAT $\dim F_\alpha \leq n$ FOR ALL α .

THEN $\dim X \leq n$.

PROOF

FIRST A LEMMA: IF F AND G ARE NON-EMPTY, $\dim F = n$, $\dim G = m$, $X = F \cup G$, LET $F \subseteq X$ BE CLOSED (WITH $\dim F \leq n$)

LET $F = (A_0, B_0), (A_1, B_1), \dots, (A_m, B_m)$ BE PAIRS OF DISJOINT CLOSED SETS IN X .

THEN THERE ARE OPEN SETS U_i, V_i SUCH THAT $A_i \subseteq U_i, B_i \subseteq V_i, \bar{U}_i \cap \bar{V}_i = \emptyset$ FOR ALL i AND $\bigcap_{i=0}^m (F \setminus (U_i \cup V_i)) = \emptyset$

PROOF TAKE PARTITIONS P_0, P_1, \dots, P_m BETWEEN $A_0 \cap F$ AND $B_0 \cap F, \dots, A_m \cap F$ AND $B_m \cap F$ IN F SUCH THAT $\bigcap_{i=0}^m P_i = \emptyset$.

LET O_i AND W_i BE OPEN IN F SUCH THAT $A_i \cap F \subseteq O_i, B_i \cap F \subseteq W_i, O_i \cap W_i = \emptyset$ AND $F \setminus P_i = O_i \cup W_i$ (CALL C_i)

NEXT LET S_0, \dots, S_m BE OPEN IN F SUCH THAT $P_i \subseteq S_i$ AND $\bar{S}_i \cap (A_i \cup B_i) = \emptyset$ (CALL D_i) SUCH $\bigcap_{i=0}^m \bar{S}_i = \emptyset$.

LET $C_i = A_i \cup O_i \setminus S_i$ AND $D_i = B_i \cup (W_i \setminus S_i)$ - C_i AND D_i ARE CLOSED

$O_i \setminus S_i = F \setminus (W_i \cup S_i)$ DIRTO $W_i \setminus S_i$

- $C_i \cap D_i = \emptyset$

LET U_i AND V_i BE OPEN SETS SUCH THAT $C_i \subseteq U_i, D_i \subseteq V_i, \bar{U}_i \cap \bar{V}_i = \emptyset$ FOR ALL i THEN $F \setminus (U_i \cup V_i) \subseteq S_i$ FOR ALL i

SO $\bigcap_{i=0}^m (F \setminus (U_i \cup V_i)) = \emptyset$ \square



NOW THE MAIN PROOF. GIVEN $\{A_i, B_i\} : i=0, \dots, n$

FIND, FOR EACH A_i , PAIRS OF OPEN SETS (U_{A_i}, V_{A_i}) , \dots , (U_{A_n}, V_{A_n})

SUCH THAT

- $A_i \subseteq U_{A_i}$, $B_i \subseteq V_{A_i}$, $\overline{U_{A_i}} \cap \overline{V_{A_i}} = \emptyset$
- $\bigcap_{i=0}^n (F_A \setminus (U_{A_i} \cup V_{A_i})) = \emptyset$
- $\overline{U_{A_i}} \subseteq U_{A_{i+1}}$, $\overline{V_{A_i}} \subseteq V_{A_{i+1}}$

TO START APPLY THE LEMMA TO

$\{A_i, B_i\} : i=0, \dots, n$ AND F_A

TO GET $(U_{1,0}, V_{1,0}), \dots, (U_{1,n}, V_{1,n})$

THEN APPLY THE LEMMA TO

$\{\overline{U_{1,i}}, \overline{V_{1,i}}\} : i=0, \dots, n$ AND $F_A \setminus \{U_{1,i}\}$

TO GET $(U_{2,0}, V_{2,0}), \dots, (U_{2,n}, V_{2,n})$

IN GENERAL APPLY IT TO

$\{U_{k-1,i}, V_{k-1,i}\} : i=0, \dots, n$ AND F_{A+1}

TO GET $(U_{k,0}, V_{k,0}), \dots, (U_{k,n}, V_{k,n})$

THEN LET $U_i = \bigcup_k U_{k,i}$, $V_i = \bigcup_k V_{k,i}$

THEN $A_i \subseteq U_i$, $B_i \subseteq V_i$

- U_i AND V_i ARE OPEN

- $U_i \cap V_i = \emptyset$

SO $P_i = X \setminus (U_i \cup V_i)$ IS A PARTITION BETWEEN A_i AND B_i

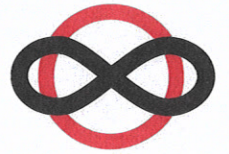
FINALLY:

$$\bigcap_{i=0}^n P_i = \emptyset.$$

$$\text{FOR: } F_A \cap P_i \subseteq F_A \setminus (U_{A_i} \cup V_{A_i}) \quad (\text{ALL } i)$$

SO

$$F_A \cap \bigcap_{i=0}^n P_i = \emptyset \quad (\text{ALL } A)$$



WE FIND $\dim \mathbb{R}^n \leq \dim [0,1]^n$

BECAUSE

$$\mathbb{R}^n = \bigcup_{k=1}^{\infty} [-k, k]^n$$

AND $\dim [-k, k]^n = \dim [0,1]^n$ FOR ALL k .

WE MUST SHOW $\dim [0,1]^n \leq n$

(THIS WILL BE (RELATIVELY) EASY

AND $\dim [0,1]^n \neq n-1$

THAT MEANS: FIND PAIRS

$(A_i, B_i) \dots, (A_n, B_n)$

OF DISJOINT CLOSED SETS

SUCH THAT FOR ALL CHOICES

OF PARTITIONS P_1, P_2, \dots, P_n WE HAVE

$$\bigcap_{i=1}^n P_i \neq \emptyset.$$

THE PAIRS WILL BE THE OBVIOUS ONES

$$A_i = \{x \in [0,1]^n : x_i = 0\}$$

$$B_i = \{x \in [0,1]^n : x_i = 1\}$$

THIS WILL BE QUITE INVOLVED,

AND USE BROWWER'S FIXED-POINT THEOREM.