OUR GOAL: PROVE THAT $\dim [0,1]^n = n$
AND $\dim \mathbb{R}^n = n$.

FIRST: $\dim [0,1]^n = \dim \mathbb{R}^n$
THIS MEANS WE CAN THEN CONCENTRATE ON $[0,1]$
WHICH IS COMPACT.

LEMMA: IF F IS A CLOSED SUBSPACE OF X
THEN $\dim F \leq \dim X$. (SO $\dim [0,1]^n \leq \dim \mathbb{R}^n$)
PROOF WE SHOW: IF $\dim X \leq n$ THEN $\dim F \leq n$.
(THIS IS HOW MANY PROOFS IN DIMENSION THEORY
WORK.)
LET $\{(A_0,B_0),\ldots,(A_n,B_n)\}$ BE A FAMILY OF
$n+1$ PAIRS OF DISJOINT CLOSED SETS IN F.
THEN $\{(A_0,B_0),\ldots,(A_n,B_n)\}$ IS ALSO SUCH A
FAMILY IN X.
IN X THERE ARE PARTITIONS $P_0,\ldots,P_n$
(P_i BETWEEN A_i AND B_i) SUCH THAT
$P_0 \cap \ldots \cap P_n = \emptyset$.
SO $X \setminus P_i = U_i \cup V_i$ WITH U_i, V_i OPEN AND
DISJOINT AND $A_i \subseteq U_i$ AND $B_i \subseteq V_i$.
INTERSECT EVERYTHING WITH F:
$Q_i = P_i \cap F$ IS A PARTITION IN F BETWEEN
A_i AND B_i: $A_i \subseteq U_i \cap F$, $B_i \subseteq V_i \cap F$
AND, OF COURSE, $Q_0 \cap \cdots \cap Q_n = \emptyset$.

To prove $\dim \mathbb{R}^n \leq \dim [0,1]^n$ requires a bit more work.

Lemma If $\{F_0, F_1, \dots, F_n\}$ is a family of closed sets with $F_0 \cap F_1 \cap \dots \cap F_n = \emptyset$ then there are open sets $O_0, O_1, \dots, O_n$ with $\overline{O_0} \cap \overline{O_1} \cap \dots \cap \overline{O_n} = \emptyset$.

Proof by induction

$n = 1$ $\quad F_0 \cap F_1 = \emptyset$

Use the metric and define

$$f : X \to \mathbb{R} \text{ by } f(x) = \frac{d(x, F_0)}{d(x, F_0) + d(x, F_1)}$$

Then:

- f is continuous (exercise)
- if $x \in F_0$ then $f(x) = 0$
- if $x \in F_1$ then $f(x) = 1$

Let $O_0 = f^{-1}[[0, 1/3)]$ and $O_1 = f^{-1}[(2/3, 1]]$

Then $\overline{O_0} \cap \overline{O_1} \subseteq f^{-1}[[0, \frac{1}{3}]] \cap f^{-1}[[2/3, 1]] = \emptyset$.

$n \to n+1$ Let $F_0, \dots, F_n, F_{n+1}$ be given

Apply the case $n = 1$ to

$G = F_0 \cap \dots \cap F_n$ and F_{n+1}

Let U and V be open with

$G \subseteq U$, $F_{n+1} \subseteq V$ and $\overline{U} \cap \overline{V} = \emptyset$

Apply the inductive hypothesis to

$F_0 \setminus U, \dots, F_n \setminus U$

to get open sets $O_0, \dots, O_n$ with

$F_i \setminus U \subseteq O_i$ (all i) and $O_0 \cap \dots \cap O_n = \emptyset$

Let $U_i = O_i \cup U$ $\quad i \leq n$

$U_{n+1} = V$

Assume $x \in \overline{U}_0 \cap \dots \cap \overline{U}_n \cap \overline{U}_{n+1}$

then $x \notin \overline{O}_0 \cap \dots \cap \overline{O}_n$, so $x \in \overline{U}$

but then $x \notin \overline{V}$

we find $\overline{U}_0 \cap \dots \cap \overline{U}_n \cap \overline{U}_{n+1} = \emptyset$.

THE COUNTABLE-CLOSED-SUM THEOREM

Assume X is the union of countably many closed sets $\{F_k : k \in \mathbb{N}\}$ such that $\dim F_k \leq n$ for all k.

Then $\dim X \leq n$.

PROOF

First a lemma

Let $F \subseteq X$ be closed with $\dim F \leq n$

Let $(A_0, B_0), \ldots, (A_n, B_n)$, be pairs of disjoint closed sets in X.

Then there are open sets $U_0, V_0, \ldots, U_n, V_n, U_{n+1}, V_{n+1}$ such that

$A_i \subseteq U_i$, $B_i \subseteq V_i$, $\overline{U_i} \cap \overline{V_i} = \emptyset$ for all i

and $\bigcap_{i=0}^{n} (F \setminus (U_i \cup V_i)) = \emptyset$

Proof Take partitions $P_0, P_1, \ldots, P_n$ between $A_0 \cap F$ and $B_0 \cap F$, ..., $A_n \cap F$ and $B_n \cap F$ in F such that $\bigcap_{i=0}^{n} P_i = \emptyset$.

Let O_i and W_i be open in F such that $A_i \cap F \subseteq O_i$, $B_i \cap F \subseteq W_i$, $O_i \cap W_i = \emptyset$ and $F \setminus P_i = O_i \cup W_i$ (all i)

Next let $S_0, \ldots, S_n$ be open such that $P_i \subseteq S_i$ and $\overline{S_i} \cap (A_i \cup B_i) = \emptyset$ (all i) such $\bigcap_{i=0}^{n} \overline{S_i} = \emptyset$.

Let $C_i = A_i \cup O_i \setminus S_i$ and $D_i = B_i \cup (W_i \setminus S_i)$

- C_i and D_i are closed

$O_i \setminus S_i = F \setminus (W_i \cup S_i)$ ditto $W_i \setminus S_i$

- $C_i \cap D_i = \emptyset$

Let U_i and V_i be open sets such that $C_i \subseteq U_i$, $D_i \subseteq V_i$, $\overline{U_i} \cap \overline{V_i} = \emptyset$ for all i then $F_i \setminus (U_i \cup V_i) \subseteq S_i$ for all i

so $\bigcap_{i=0}^{n} (F \setminus (U_i \cup V_i)) = \emptyset$ $\square$

NOW THE MAIN PROOF. GIVEN $\{(A_i, B_i) : i = 0, \ldots, n\}$
FIND, FOR EACH k, PAIRS OF OPEN SETS
$(U_{k,0}, V_{k,0}), \ldots, (U_{k,n}, V_{k,n})$
SUCH THAT
- $A_i \subseteq U_{k,i}$, $B_i \subseteq V_{k,i}$, $\overline{U}_{k,i} \cap \overline{V}_{k,i} = \emptyset$
- $\bigcap_{i=0}^{n} (F_k \setminus (U_{k,i} \cup V_{k,i})) = \emptyset$
- $\overline{U}_{k,i} \subseteq U_{k+1,i}$, $\overline{V}_{k,i} \subseteq V_{k+1,i}$

TO START APPLY THE LEMMA TO
$\{(A_i, B_i) : i \leq n\}$ AND F_1
TO GET $(U_{1,0}, V_{1,0}), \ldots, (U_{1,n}, V_{1,n})$
THEN APPLY THE LEMMA TO
$\{(\overline{U}_{1,i}, \overline{V}_{1,i}) : i \leq n\}$ AND F_2
TO GET $(U_{2,0}, V_{2,0}), \ldots, (U_{2,n}, V_{2,n})$
IN GENERAL APPLY IT TO
$\{(U_{k,i}, V_{k,i}) : i \leq n\}$ AND F_{k+1}
TO GET $(U_{k+1,0}, V_{k+1,0}), \ldots, (U_{k+1,n}, V_{k+1,n})$

THEN LET $U_i = \bigcup_k U_{k,i}$, $V_i = \bigcup_k V_{k,i}$
THEN
- $A_i \subseteq U_i$, $B_i \subseteq V_i$
- U_i AND V_i ARE OPEN
- $U_i \cap V_i = \emptyset$

SO $P_i = X \setminus (U_i \cup V_i)$ IS A PARTITION BETWEEN A_i AND B_i

FINALLY:
$$\bigcap_{i=0}^{n} P_i = \emptyset.$$
FOR: $F_k \cap P_i \subseteq F_k \setminus (U_{k,i} \cup V_{k,i})$ (ALL i)
SO
$$F_k \cap \bigcap_{i=0}^{n} P_i = \emptyset \qquad \text{(ALL } k\text{)}$$

WE FIND $\dim \mathbb{R}^n \le \dim [0,1]^n$

BECAUSE

$$\mathbb{R}^n = \bigcup_{k=1}^{\infty} [-k,k]^n$$

AND $\dim [-k,k]^n = \dim [0,1]^n$ FOR ALL k.

WE MUST SHOW $\dim [0,1]^n \le n$

(THIS WILL BE (RELATIVELY) EASY

AND $\dim [0,1]^n \not\le n-1$

THAT MEANS: FIND PAIRS

$(A_1, B_1), \ldots, (A_n, B_n)$

OF DISJOINT CLOSED SETS

SUCH THAT FOR <u>ALL</u> CHOICES

OF PARTITIONS $P_1, P_2, \ldots, P_n$ WE HAVE

$$\bigcap_{i=1}^{n} P_i \neq \emptyset.$$

THE PAIRS WILL BE THE OBVIOUS ONES

$$A_i = \{x \in [0,1]^n : x_i = 0\}$$
$$B_i = \{x \in [0,1]^n : x_i = 1\}$$

THIS WILL BE QUITE INVOLVED,

AND USE BROUWER'S FIXED-POINT THEOREM.