THE COUNTABLE-CLOSED-SUM THEOREM

ASSUME X IS THE UNION OF COUNTABLY MANY CLOSED SETS $\{F_k : k \in \mathbb{N}\}$ SUCH THAT $\dim F_k \leq n$ FOR ALL k.

THEN $\dim X \leq n$.

PROOF

FIRST A LEMMA

LET $F \subseteq X$ BE CLOSED WITH $\dim F \leq n$

LET $(A_0, B_0), \ldots, (A_n, B_n)$, BE PAIRS OF DISJOINT CLOSED SETS IN X.

THEN THERE ARE OPEN SETS $U_0, V_0, \ldots, U_n, V_n, U_{n+1}, V_{n+1}$ SUCH THAT

$A_i \subseteq U_i$, $B_i \subseteq V_i$, $\overline{U_i} \cap \overline{V_i} = \emptyset$ FOR ALL i

AND $\bigcap_{i=0}^{n} (F \setminus (U_i \cup V_i)) = \emptyset$

<u>PROOF</u> TAKE PARTITIONS $P_0, P_1, \ldots, P_n$ BETWEEN $A_0 \cap F$ AND $B_0 \cap F$, ..., $A_n \cap F$ AND $B_n \cap F$ IN F SUCH THAT $\bigcap_{i=0}^{n} P_i = \emptyset$.

LET O_i AND W_i BE OPEN <u>IN F</u> SUCH THAT $A_i \cap F \subseteq O_i$, $B_i \cap F \subseteq W_i$, $O_i \cap W_i = \emptyset$ AND $F \setminus P_i = O_i \cup W_i$ (ALL i)

NEXT LET $S_0, \ldots, S_n$ BE OPEN SUCH THAT $P_i \subseteq S_i$ AND $\overline{S_i} \cap (A_i \cup B_i) = \emptyset$ (ALL i)

$\bigcap_{i=0}^{n} \overline{S_i} = \emptyset$.

LET $C_i = A_i \cup O_i \setminus S_i$ AND $D_i = B_i \cup (W_i \setminus S_i)$

- C_i AND D_i ARE CLOSED

$O_i \setminus S_i = F \setminus (W_i \cup S_i)$ DITTO $W_i \setminus S_i$

- $C_i \cap D_i = \emptyset$

LET U_i AND V_i BE OPEN SETS SUCH THAT $C_i \subseteq U_i$, $D_i \subseteq V_i$, $\overline{U_i} \cap \overline{V_i} = \emptyset$ FOR ALL i THEN $F_i \setminus (U_i \cup V_i) \subseteq S_i$ FOR ALL i

SO $\bigcap_{i=0}^{n} (F \setminus (U_i \cup V_i)) = \emptyset$ $\square$

NOW THE MAIN PROOF. GIVEN $\{(A_i, B_i) : i = 0, \dots, n\}$

FIND, FOR EACH k, PAIRS OF OPEN SETS

$(U_{k,0}, V_{k,0}), \dots, (U_{k,n}, V_{k,n})$

SUCH THAT

- $A_i \subseteq U_{k,i}$, $B_i \subseteq V_{k,i}$, $\overline{U_{k,i}} \cap \overline{V_{k,i}} = \emptyset$
- $\bigcap_{i=0}^{n} (F_k \setminus (U_{k,i} \cup V_{k,i})) = \emptyset$
- $\overline{U_{k,i}} \subseteq U_{k+1,i}$, $\overline{V_{k,i}} \subseteq V_{k+1,i}$

TO START APPLY THE LEMMA TO
$\{(A_i, B_i) : i \le n\}$ AND F_1
TO GET $(U_{1,0}, V_{1,0}), \dots, (U_{1,n}, V_{1,n})$

THEN APPLY THE LEMMA TO
$\{(\overline{U_{1,i}}, \overline{V_{1,i}}) : i \le n\}$ AND F_2
TO GET $(U_{2,0}, V_{2,0}), \dots, (U_{2,n}, V_{2,n})$

IN GENERAL APPLY IT TO
$\{(U_{k,i}, V_{k,i}) : i \le n\}$ AND F_{k+1}
TO GET $(U_{k+1,0}, V_{k+1,0}), \dots, (U_{k+1,n}, V_{k+1,n})$

THEN LET $U_i = \bigcup_k U_{k,i}$, $V_i = \bigcup_k V_{k,i}$

THEN
- $A_i \subseteq U_i$, $B_i \subseteq V_i$
- U_i AND V_i ARE OPEN
- $U_i \cap V_i = \emptyset$

SO $P_i = X \setminus (U_i \cup V_i)$ IS A PARTITION BETWEEN A_i AND B_i

FINALLY:

$$\bigcap_{i=0}^{n} P_i = \emptyset.$$

FOR: $F_k \cap P_i \subseteq F_k \setminus (U_{k,i} \cup V_{k,i})$ (ALL i)

SO

$$F_k \cap \bigcap_{i=0}^{n} P_i = \emptyset \qquad \text{(ALL } k\text{)}$$

WE FIND $\dim \mathbb{R}^n \leq \dim [0,1]^n$
BECAUSE
$$\mathbb{R}^n = \bigcup_{k=1}^{\infty} [-k,k]^n$$
AND $\dim [-k,k]^n = \dim [0,1]^n$ FOR ALL k.

WE MUST SHOW $\dim [0,1]^n \leq n$
(THIS WILL BE (RELATIVELY) EASY
AND $\dim [0,1]^n \nleq n-1$
THAT MEANS: FIND PAIRS
$(A_1, B_1), \ldots, (A_n, B_n)$
OF DISJOINT CLOSED SETS
SUCH THAT FOR <u>ALL</u> CHOICES
OF PARTITIONS $P_1, P_2, \ldots, P_n$ WE HAVE
$$\bigcap_{i=1}^{n} P_i \neq \emptyset.$$
THE PAIRS WILL BE THE OBVIOUS ONES
$$A_i = \{x \in [0,1]^n : x_i = 0\}$$
$$B_i = \{x \in [0,1]^n : x_i = 1\}$$
THIS WILL BE QUITE INVOLVED,
AND USE BROUWER'S FIXED-POINT THEOREM.

THEOREM $\dim [0,1]^n \leq n$.

PROOF

LET $(A_0, B_0), \ldots, (A_n, B_n)$ A FAMILY OF $n+1$ PAIRS OF DISJOINT CLOSED SETS.

WE FIND PARTITIONS $P_0, \ldots, P_n$ WITH $\bigcap_{i=0}^{n} P_i = \emptyset$.

FOR $k \in \mathbb{N} \cup \{0\}$ LET $\mathbb{Q}_k = \mathbb{Q} + k\sqrt{2}$

NOTE: . EACH $\mathbb{Q}_k$ IS DENSE IN $\mathbb{R}$

. IF $k \neq l$ THEN $\mathbb{Q}_k \cap \mathbb{Q}_l = \emptyset$.

LET $k \in \{0, \ldots, n\}$

FOR EACH $x \in A_k$ TAKE

$p_1^x, q_1^x, \ldots, p_n^x, q_n^x \in \mathbb{Q}_k$

SUCH THAT

- $x \in U_x = \prod_{i=1}^{n} (p_i^x, q_i^x)$
- $\prod_{i=1}^{n} [p_i^x, q_i^x] \cap B_k = \emptyset$.

NOTE: IF $y \in \overline{U}_x \setminus U_x$ THEN AT LEAST ONE COORDINATE OF y IS IN $\mathbb{Q}_k$

INDEED THE BOUNDARY OF U_x IS EQUAL TO $\{p_1^x, q_1^x\} \times [0,1]^{n-1} \cup [0,1] \times \{p_2^x, q_2^x\} \times [0,1]^{n-2} \cup \ldots$

$\ldots [0,1]^{n-1} \times \{p_n^x, q_n^x\}$ (DRAW A PICTURE)

BY COMPACTNESS LET $F_k \subseteq A_k$ BE FINITE SUCH THAT

$$A_k \subseteq U_k = \bigcup_{x \in F_k} U_x.$$

LET $V_k = [0,1]^n \setminus \overline{U}_k$

THEN V_k IS OPEN AND $B_k \subseteq V_k$

LET $P_k = [0,1]^n \setminus (U_k, V_k)$

THEN P_k IS JUST THE BOUNDARY OF U_k

AND $P_k \subseteq \bigcup_{x \in F_k} (\overline{U}_x \setminus U_x)$ (EXERCISE)

SO, IF $y \in P_k$ THEN AT LEAST ONE COORDINATE OF y IS IN $\mathbb{Q}_k$.

WE HAVE CONSTRUCTED FOR $k \in \{0, \dots, n\}$ A PARTITION P_k BETWEEN A_k AND B_k.
WE CLAIM THAT

$$P_0 \cap \dots \cap P_n = \varphi$$

INDEED:

IF $y \in P_0 \cap \dots \cap P_n$
THEN y HAS A COORDINATE IN $\mathcal{Q}_0$, IN $\mathcal{Q}_1, \dots,$ IN $\mathcal{Q}_n$.
BUT THE $\mathcal{Q}_k$ ARE DISJOINT, SO y MUST HAVE $n+1$ COORDINATES, WHICH DOES NOT WORK IN $[0,1]^n$.

EXERCISE

PROVE: $\dim X \le n$ IF AND ONLY IF
FOR EVERY FAMILY OF $n+2$ CLOSED SETS $F_0, F_1, \dots, F_n, F_{n+1}$ WITH $\bigcap_{i=0}^{n+1} F_i = \varnothing$
THERE IS A FAMILY $G_0, \dots, G_n, G_{n+1}$ OF $n+2$ CLOSED SETS WITH

- $F_i \subseteq G_i$ FOR ALL i
- $\bigcap_{i=0}^{n+1} G_i = \varnothing$
- $\bigcup_{i=0}^{n} G_i = X$

IF AND ONLY IF (DUALLY)
FOR EVERY FAMILY $\{U_0, U_1, \dots, U_n, U_{n+1}\}$ OF $n+2$ OPEN SETS WITH $\bigcup_{i=0}^{n+1} U_n = X$
THERE IS A FAMILY $V_0, \dots, V_n, V_{n+1}$ OF $n+2$ OPEN SETS SUCH THAT

- $V_i \subseteq U_i$ FOR ALL i
- $\bigcup_{i=0}^{n+1} V_i = X$
- $\bigcap_{i=0}^{n+1} V_i = \varnothing$.

TOWARD PROVING THAT $\dim [0,1]^n \geq n$.

WE PLAN TO SHOW THAT THE SIDES OF THE n-CUBE WITNESS THIS:

(*) FOR $i \leq n$ LET $A_i = \{x \in [0,1]^n : x_i = 0\}$
AND $B_i = \{x \in [0,1]^n : x_i = 1\}$
THEN FOR EVERY FAMILY $\{P_1, P_2, \dots, P_n\}$
WHERE P_i IS A PARTITION BETWEEN A_i AND B_i
WE HAVE $\bigcap_{i=1}^n P_i \neq \emptyset$.

AN INTERESTING OBSERVATION IS THAT THIS IS SOMEHOW NECESSARY.
ASSUME THERE IS A METRIC SPACE X WITH $\dim X \geq n$. THEN STATEMENT (*) HOLDS.
TO SEE THIS LET $(F_1, G_1), \dots, (F_n, G_n)$ BE PAIRS OF DISJOINT CLOSED SETS SUCH THAT $\bigcap_{i=1}^n L_i \neq \emptyset$ WHENEVER L_i IS A PARTITION BETWEEN F_i AND G_i FOR $i = 1, \dots, n$.
DEFINE $f_i : X \to [0,1]$ BY

$$f_i(x) = \frac{d(x, F_i)}{d(x, F_i) + d(x, G_i)} \qquad (i = 1, \dots, n)$$

AND $f : X \to [0,1]^n$ BY $f(x) = (f_1(x), \dots, f_n(x))$

- f IS CONTINUOUS (EXERCISE)
- IF P_i IS A PARTITION BETWEEN A_i AND B_i THEN $f^{-1}[P_i]$ IS A PARTITION BETWEEN F_i AND G_i

SAY $A_i \subseteq U_i$, $B_i \subseteq V_i$, $U_i \cap V_i = \emptyset$ AND
$[0,1]^n \setminus P_i = U_i \cup V_i$.
THEN $X \setminus f^{-1}[P_i] = f^{-1}[U_i] \cup f^{-1}[V_i]$
AND $f^{-1}[U_i] \cap f^{-1}[V_i] = \emptyset$

ALSO IF $x \in F_i$ THEN $f_i(x) = 0$
SO $f(x) \in A_i$; THIS IMPLIES
THAT $F_i \subseteq f^{-1}[U_i]$
AND LIKEWISE $G_i \subseteq f^{-1}[V_i]$

- SO, IF WE HAVE PARTITIONS $P_1, \ldots, P_n$ BETWEEN A_1 AND B_1, ..., A_n AND B_n THEN FOR ALL i THE PREIMAGE $f^{-1}[P_i]$ IS A PARTITION BETWEEN F_i AND G_i

BUT THEN $\bigcap_{i=1}^{n} f^{-1}[P_i] \neq \emptyset$
BUT $\bigcap_{i=1}^{n} f^{-1}[P_i] = f^{-1}[\bigcap_{i=1}^{n} P_i]$
SO $\bigcap_{i=1}^{n} P_i \neq \emptyset$.

So ⊛ IS ESSENTIALLY THE ONLY WAY TO SHOW THAT $\dim [0,1]^n \geq n$.

How DO WE PROVE ⊛?

WE LOOK FOR SOME EQUIVALENT STATEMENTS AND HOPE FOR ONE THAT WE CAN PROVE.

(1) IF $f : [0,1]^n \to [0,1]^n$ IS CONTINUOUS THEN THERE IS AN $x \in [0,1]^n$ SUCH THAT $f(x) = x$
[BROUWER'S FIXED-POINT THEOREM]

(2) THERE IS NO CONTINUOUS MAP $f : B^n \to B^n$ SUCH THAT $f[B_n] \subseteq S^{n-1}$ AND $f(x) = x$ FOR ALL $x \in S^{n-1}$ [NO-RETRACTION THEOREM]

- $B^n = \{ x \in \mathbb{R}^n : \|x\| \leq 1 \}$
- $S^{n-1} = \{ x \in \mathbb{R}^n : \|x\| = 1 \}$.