

THE COUNTABLE-CLOSED-SUM THEOREM

ASSUME X IS THE UNION OF COUNTABLY MANY CLOSED SETS $\{F_\alpha : \alpha \in \mathbb{N}\}$ SUCH THAT $\dim F_\alpha \leq n$ FOR ALL α .

THEN $\dim X \leq n$.

PROOF

FIRST A LEMMA: IF F AND G ARE NON-EMPTY, $\dim F \leq n$ AND $\dim G \leq n$, THEN $F \cup G$ IS CLOSED (WITH $\dim F \cup G \leq n$).

LET $F \in X$ BE CLOSED (WITH $\dim F \leq n$).

LET $F = (A_0, B_0), (A_1, B_1), \dots, (A_m, B_m)$ BE PAIRS OF DISJOINT CLOSED SETS IN F .

THEN THERE ARE OPEN SETS $U_0, V_0, \dots, U_m, V_m$ SUCH THAT

$A_i \subseteq U_i, B_i \subseteq V_i, \bar{U}_i \cap \bar{V}_i = \emptyset$ FOR ALL i .

AND $\bigcap_{i=0}^m (F \setminus (U_i \cup V_i)) = \emptyset$.

PROOF: TAKE PARTITIONS P_0, P_1, \dots, P_m BETWEEN $A_0 \cap F$ AND $B_0 \cap F, \dots, A_m \cap F$ AND $B_m \cap F$ IN F SUCH THAT $\bigcap_{i=0}^m P_i = \emptyset$.

LET O_i AND W_i BE OPEN IN F SUCH THAT $A_i \cap F \subseteq O_i, B_i \cap F \subseteq W_i, O_i \cap W_i = \emptyset$ AND $F \setminus P_i = O_i \cup W_i$ (CALL i).

NEXT LET S_0, \dots, S_m BE OPEN IN F SUCH THAT $P_i \subseteq S_i$ AND $\bar{S}_i \cap (A_i \cup B_i) = \emptyset$ (CALL i).

SUCH $\bigcap_{i=0}^m \bar{S}_i = \emptyset$.

LET $C_i = A_i \cup O_i \setminus S_i$ AND $D_i = B_i \cup (W_i \setminus S_i)$.

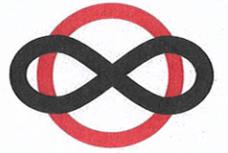
C_i AND D_i ARE CLOSED.

$O_i \setminus S_i = F \setminus (W_i \cup S_i)$ DITTO $W_i \setminus S_i$.

$C_i \cap D_i = \emptyset$.

LET U_i AND V_i BE OPEN SETS SUCH THAT $C_i \subseteq U_i, D_i \subseteq V_i, \bar{U}_i \cap \bar{V}_i = \emptyset$ FOR ALL i . THEN $F \setminus (U_i \cup V_i) \subseteq S_i$ FOR ALL i .

SO $\bigcap_{i=0}^m (F \setminus (U_i \cup V_i)) = \emptyset$. \square



NOW THE MAIN PROOF. GIVEN $\{ (A_i, B_i) : i=0, \dots, n \}$

FIND, FOR EACH A , PAIRS OF OPEN SETS

$(U_{A,0}, V_{A,0}), \dots, (U_{A,n}, V_{A,n})$

SUCH THAT

- $A_i \subseteq U_{A,i}, B_i \subseteq V_{A,i}, \overline{U_{A,i}} \cap \overline{V_{A,i}} = \emptyset$

- $\bigcap_{i=0}^n (F_A \setminus (U_{A,i} \cup V_{A,i})) = \emptyset$

- $\overline{U_{A,i}} \subseteq U_{A,i+1}, \overline{V_{A,i}} \subseteq V_{A,i+1}$

TO START APPLY THE LEMMA TO

$\{ (A_i, B_i) : i \leq n \}$ AND F_A

TO GET $(U_{1,0}, V_{1,0}), \dots, (U_{1,n}, V_{1,n})$

THEN APPLY THE LEMMA TO

$\{ (\overline{U_{1,i}}, \overline{V_{1,i}}) : i \leq n \}$ AND $F_A \setminus \{U_{1,0}\}$

TO GET $(U_{2,0}, V_{2,0}), \dots, (U_{2,n}, V_{2,n})$

IN GENERAL APPLY IT TO

$\{ (U_{A,i}, V_{A,i}) : i \leq n \}$ AND F_{A+1}

TO GET $(U_{A+1,0}, V_{A+1,0}), \dots, (U_{A+1,n}, V_{A+1,n})$

THEN LET $U_i = \bigcup_{A \in \mathcal{A}} U_{A,i}, V_i = \bigcup_{A \in \mathcal{A}} V_{A,i}$

THEN $A_i \subseteq U_i, B_i \subseteq V_i$

- U_i AND V_i ARE OPEN

- $U_i \cap V_i = \emptyset$

SO $P_i = X \setminus (U_i \cup V_i)$ IS A PARTITION

BETWEEN A_i AND B_i

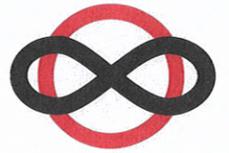
FINALLY:

$$\bigcap_{i=0}^n P_i = \emptyset.$$

FOR: $F_A \cap P_i \subseteq F_A \setminus (U_{A,i} \cup V_{A,i})$ (ALL A)

SO

$$F_A \cap \bigcap_{i=0}^n P_i = \emptyset \quad (\text{ALL } A)$$



WE FIND $\dim \mathbb{R}^n \leq \dim [0,1]^n$

BECAUSE

$$\mathbb{R}^n = \bigcup_{k=1}^{\infty} [-k, k]^n$$

AND $\dim [-k, k]^n = \dim [0,1]^n$ FOR ALL k .

WE MUST SHOW $\dim [0,1]^n \leq n$

(THIS WILL BE (RELATIVELY) EASY

AND $\dim [0,1]^n \neq n-1$

THAT MEANS: FIND PAIRS

$(A_i, B_i) \dots (A_n, B_n)$

OF DISJOINT CLOSED SETS

SUCH THAT FOR ALL CHOICES

OF PARTITIONS P_1, P_2, \dots, P_n WE HAVE

$$\bigcap_{i=1}^n P_i \neq \emptyset.$$

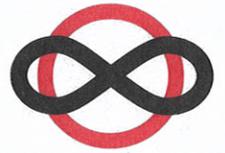
THE PAIRS WILL BE THE OBVIOUS ONES

$$A_i = \{x \in [0,1]^n : x_i = 0\}$$

$$B_i = \{x \in [0,1]^n : x_i = 1\}$$

THIS WILL BE QUITE INVOLVED,

AND USE BROWWER'S FIXED-POINT THEOREM.



THEOREM $\dim [0, 1]^n \leq n$

PROOF

LET $(A_0, B_0), \dots, (A_n, B_n)$ A FAMILY OF $n+1$ PAIRS OF DISJOINT CLOSED SETS.

WE FIND PARTITIONS P_0, \dots, P_n WITH

$$\bigcap_{i=0}^n P_i = \emptyset.$$

FOR $R \in \mathbb{N} \cup \{0\}$ LET $\mathcal{Q}_R = \mathcal{Q} + R\sqrt{2}$

NOTE: EACH \mathcal{Q}_R IS DENSE IN \mathbb{R}

IF $R \neq L$ THEN $\mathcal{Q}_R \cap \mathcal{Q}_L = \emptyset$.

LET $k \in \{0, \dots, n\}$

FOR EACH $x \in A_k$ TAKE

$$p_1^x, q_1^x, \dots, p_n^x, q_n^x \in \mathcal{Q}_k$$

SUCH THAT

$$x \in U_x = \prod_{i=1}^n (p_i^x, q_i^x)$$

$$\prod_{i=1}^n [p_i^x, q_i^x] \cap B_k = \emptyset.$$

NOTE: IF $y \in \bar{U}_x \setminus U_x$ THEN AT LEAST

ONE COORDINATE OF y IS IN \mathcal{Q}_k

INDEED THE BOUNDARY OF U_x IS EQUAL

$$\text{TO } \{p_1^x, q_1^x\} \times [0, 1]^{n-1} \cup [0, 1] \times \{p_2^x, q_2^x\} \times [0, 1]^{n-2} \cup \dots$$

$$\dots [0, 1]^{n-1} \times \{p_n^x, q_n^x\} \quad (\text{DRAW A PICTURE})$$

By compactness LET $F_k \subseteq A_k$ BE FINITE

SUCH THAT

$$A_k \subseteq U_k = \bigcup_{x \in F_k} U_x.$$

$$\text{LET } V_k = [0, 1]^n \setminus \bar{U}_k$$

THEN V_k IS OPEN AND $B_k \subseteq V_k$

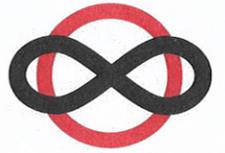
$$\text{LET } P_k = [0, 1]^n \setminus (U_k, V_k)$$

THEN P_k IS JUST THE BOUNDARY OF U_k

AND $P_k \subseteq \bigcup_{x \in F_k} (\bar{U}_x \setminus U_x)$ (EXERCISE)

SO, IF $y \in P_k$ THEN AT LEAST ONE

COORDINATE OF y IS IN \mathcal{Q}_k .



WE HAVE CONSTRUCTED FOR $\mathbb{R} \in (0, \dots, n)$
 A PARTITION P_R BETWEEN A_R AND B_R .
 WE CLAIM THAT

$$P_0 \cap \dots \cap P_n = \emptyset$$

INDEED:

IF $y \in P_0 \cap \dots \cap P_n$

THEN y HAS A COORDINATE

IN $Q_0, \dots, \text{IN } Q_1, \dots, \text{IN } Q_n$

BUT THE Q_R ARE DISJOINT, SO

y MUST HAVE $n+1$ COORDINATES,
 WHICH DOES NOT WORK IN $[0, 1]^n$.

EXERCISE

PROVE: $\dim X \leq n$ IF AND ONLY IF
 FOR EVERY FAMILY OF $n+2$ CLOSED SETS
 $F_0, F_1, \dots, F_n, F_{n+1}$ WITH $\bigcap_{i=0}^{n+1} F_i = \emptyset$
 THERE IS A FAMILY G_0, \dots, G_n, G_{n+1} OF
 $n+2$ CLOSED SETS WITH

$$- F_i \subseteq G_i \text{ FOR ALL } i$$

$$- \bigcap_{i=0}^{n+1} G_i = \emptyset$$

$$- \bigcup_{i=0}^n G_i = X$$

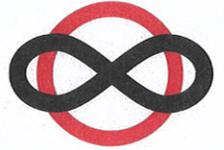
IF AND ONLY IF (DUALLY)

FOR EVERY FAMILY $\{U_0, U_1, \dots, U_n, U_{n+1}\}$
 OF $n+2$ OPEN SETS WITH $\bigcup_{i=0}^{n+1} U_i = X$
 THERE IS A FAMILY V_0, \dots, V_n, V_{n+1} OF
 $n+2$ OPEN SETS SUCH THAT

$$- V_i \subseteq U_i \text{ FOR ALL } i$$

$$- \bigcup_{i=0}^{n+1} V_i = X$$

$$- \bigcap_{i=0}^{n+1} V_i = \emptyset$$



TOWARD PROVING THAT $\dim [0,1]^m \geq m$.

WE PLAN TO SHOW THAT THE SIDES
A OF THE m -CUBE WITNESS THIS:

FOR $i \leq m$ LET $A_i = \{x \in [0,1]^m : x_i = 0\}$
AND $B_i = \{x \in [0,1]^m : x_i = 1\}$
 (*) THEN FOR EVERY FAMILY $\{P_1, P_2, \dots, P_m\}$
WHERE P_i IS A PARTITION BETWEEN A_i AND B_i
WE HAVE $\bigcap_{i=1}^m P_i \neq \emptyset$.

AN INTERESTING OBSERVATION IS THAT THIS
IS SOMEHOW NECESSARY.

ASSUME THERE IS A METRIC SPACE X WITH
 $\dim X \geq m$. THEN STATEMENT (*) HOLDS.

TO SEE THIS LET $(F_1, G_1), \dots, (F_m, G_m)$ BE
PAIRS OF DISJOINT CLOSED SETS SUCH THAT
 $\bigcap_{i=1}^m L_i \neq \emptyset$ WHENEVER L_i IS A PARTITION
BETWEEN F_i AND G_i FOR $i=1, \dots, m$

DEFINE $f_i : X \rightarrow [0,1]$ BY

$$f_i(x) = \frac{d(x, F_i)}{d(x, F_i) + d(x, G_i)} \quad (i=1, \dots, m)$$

AND $f : X \rightarrow [0,1]^m$ BY $f(x) = (f_1(x), \dots, f_m(x))$

- f IS CONTINUOUS (EXERCISE)
- IF P_i IS A PARTITION BETWEEN A_i AND B_i
THEN $f^{-1}[P_i]$ IS A PARTITION BETWEEN
 F_i AND G_i

SAFELY $A_i \in U_i, B_i \in V_i, U_i \cap V_i = \emptyset$ AND
 $[0,1]^m \setminus P_i = U_i \cup V_i$

THEN $X \setminus f^{-1}[P_i] = f^{-1}[U_i] \cup f^{-1}[V_i]$

AND $f^{-1}[U_i] \cap f^{-1}[V_i] = \emptyset$



ALSO IF $x \in F_i$ THEN $f(x) = 0$

SO $f(x) \in A_i$; THIS IMPLIES

THAT $F_i \in f^{-1}[A_i]$

AND LIKEWISE $G_i \in f^{-1}[B_i]$

• SO, IF WE HAVE PARTITIONS P_1, \dots, P_m

BETWEEN A_i AND B_1, \dots, B_m AND B_m

THEN FOR ALL i THE PREIMAGE $f^{-1}[P_i]$

IS A PARTITION BETWEEN F_i AND G_i

BUT THEN $\bigcap_{i=1}^m f^{-1}[P_i] \neq \emptyset$

BUT $\bigcap_{i=1}^m f^{-1}[P_i] = f^{-1}[\bigcap_{i=1}^m P_i]$

SO $\bigcap_{i=1}^m P_i \neq \emptyset$

So \otimes IS ESSENTIALLY THE ONLY WAY TO SHOW
THAT $\dim [0,1]^m \geq m$.

How do we prove \otimes ?

WE LOOK FOR SOME EQUIVALENT STATEMENTS
AND HOPE FOR ONE THAT WE CAN PROVE.

① IF $f: [0,1]^m \rightarrow [0,1]^m$ IS CONTINUOUS
THEN THERE IS AN $x \in [0,1]^m$
SUCH THAT $f(x) = x$
[BROUWER'S FIXED-POINT THEOREM]

② THERE IS NO CONTINUOUS MAP $f: B^n \rightarrow B^n$
SUCH THAT $f[B^n] \subseteq S^{n-1}$ AND $f(x) = x$
FOR ALL $x \in S^{n-1}$ [NO-RETRACTION THEOREM]
• $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$
• $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$