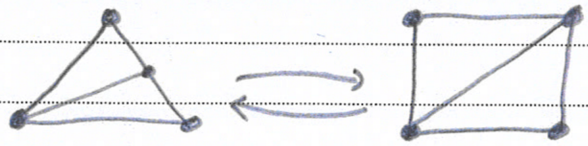
EXERCISE: EVERY 2-DIMENSIONAL SIMPLEX IS HOMEOMORPHIC WITH $[0,1]^2$.

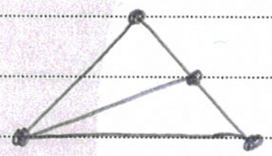
HINT



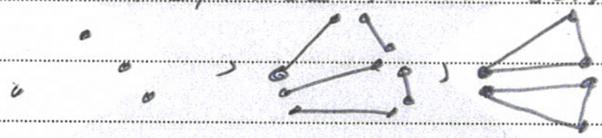
SUBDIVISIONS

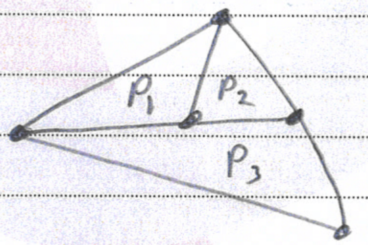
A SIMPLICIAL SUBDIVISION OF A SIMPLEX S IS A FAMILY $\mathcal{P}$ OF SIMPLEXES SUCH THAT

- $\mathcal{P}$ IS FINITE
- $S = \bigcup \mathcal{P}$
- IF $P, Q \in \mathcal{P}$ THEN $P \cap Q = \emptyset$ OR $P \cap Q$ IS A COMMON FACE OF P AND Q
- IF $P \in \mathcal{P}$ THEN ALL FACES OF P ARE IN $\mathcal{P}$



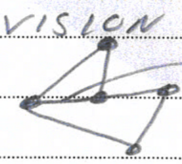
THIS REPRESENTS A SUBDIVISION:





THIS DOES NOT REPRESENT A SUBDIVISION

$P_1 \cap P_3$:



NOT A COMMON FACE, NOT A FACE OF P_3

THE MESH OF A SUBDIVISION IS

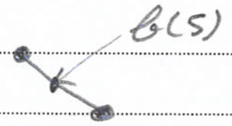
$$\max\{\operatorname{diam} P : P \in \mathcal{P}\}$$

WHERE, IN GENERAL, $\operatorname{diam} A = \sup\{d(x,y) : x, y \in A\}$

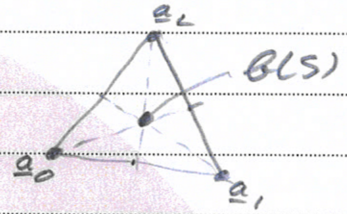
EXERCISE PROVE

$$\operatorname{diam}[a_0, \ldots, a_k] = \operatorname{diam}\{\underline{a}_0, \ldots, \underline{a}_k\}$$

THE BARYCENTER $b(S)$ OF A SIMPLEX
$S = [a_0, \dots, a_k]$ IS THE POINT
$$\frac{1}{k+1} a_0 + \frac{1}{k+1} a_1 + \dots + \frac{1}{k+1} a_k$$



$k = 1$ MIDPOINT OF $[a_0, a_1]$



$k = 2$ CENTROID OF $[a_0, a_1, a_2]$

BARYCENTRIC SUBDIVISIONS

LET $S = [a_0, \dots, a_k]$ BE A SIMPLEX
LET $P_0 \supset P_1 \supset \dots \supset P_\ell$ BE A DECREASING SEQUENCE OF FACES OF S.
THEN THE BARYCENTERS $b(P_0), b(P_1), \dots, b(P_\ell)$ ARE AFFINELY INDEPENDENT.

- EACH P_i CORRESPONDS TO A SUBSET F_i OF $\{a_0, \dots, a_k\}$ AND $F_0 \supset F_1 \supset \dots \supset F_\ell$; EXPAND THIS TO A SEQUENCE $G_0 \supset G_1 \supset \dots \supset G_k$ OF SUBSETS SUCH THAT $G_0 = \{a_{i_0}, \dots, a_{i_k}\}$, $G_1 = \{a_{i_1}, \dots, a_{i_k}\}, \dots,$ $G_k = \{a_{i_k}\}$ WHERE $j \mapsto i_j$ IS A PERMUTATION OF $\{0, 1, \dots, k\}$
- SO WLOG $\ell = k$, AND

$$b(P_0) = \frac{1}{k+1} a_{i_0} + \frac{1}{k+1} a_{i_1} + \dots + \frac{1}{k+1} a_{i_k}$$
$$b(P_1) = \frac{1}{k} a_{i_1} + \dots + \frac{1}{k} a_{i_k}$$
$$\vdots$$
$$b(P_k) = a_{i_k}$$

ASSUME $\mu_0 b(P_0) + \dots + \mu_k b(P_k) = 0$
AND $\mu_0 + \dots + \mu_k = 0$

WE GET

$$\mu_0 \frac{1}{k+1} a_{i_0} + \left(\mu_0 \frac{1}{k+1} + \mu_1 \frac{1}{k}\right) a_{i_1} + \left(\mu_0 \frac{1}{k+1} + \dots + \mu_k\right) a_{i_k} = 0.$$

AND SO $\mu_0 \frac{1}{k+1} = 0$, $\mu_0 \frac{1}{k+1} + \mu_1 \frac{1}{k} = 0$, ...,
$\mu_0 \frac{1}{k+1} + \mu_1 \frac{1}{k} + \dots + \mu_k = 0$

OR

$$\begin{pmatrix} 1/k+1 & 0 & 0 & & 0 \\ 1/k+1 & 1/k & 0 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ & & & & 0 \\ 1/k+1 & 1/k & & & 1 \end{pmatrix} \begin{pmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \\ \mu_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix}$$

AND SO $\mu_0 = \mu_1 = \dots = \mu_k = 0$

- THE SIMPLEX $[b(P_0) \dots b(P_k)]$ IS EQUAL TO THE SET

$$\{ x \in S : \lambda_{i_0}(x) \le \lambda_{i_1}(x) \le \dots \le \lambda_{i_k}(x) \}$$

NOTE IN THE PREVIOUS POINT THE μ_i AND λ_i IN

$$\mu_0 b(P_0) + \mu_1 b(P_1) + \dots + \mu_k (P_k) = \lambda_0 a_0 + \lambda_1 a_1 + \dots + \lambda_k a_k$$

SATISFY

$$\begin{pmatrix} \frac{1}{k+1} & 0 & 0 & \dots & 0 \\ \frac{1}{k+1} & \frac{1}{k} & & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ \frac{1}{k+1} & \frac{1}{k} & & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_k \end{pmatrix} = \begin{pmatrix} \lambda_{i_0} \\ \lambda_{i_1} \\ \vdots \\ \lambda_{i_k} \end{pmatrix}$$

OR FROM THE BOTTOM TO THE TOP

$1 \cdot \mu_k = \lambda_{i_k} - \lambda_{i_{k-1}}$, $\frac{1}{2}\mu_{k-1} = \lambda_{i_{k-1}} - \lambda_{i_{k-2}}$, ...
..., $\frac{1}{k}\mu_1 = \lambda_{i_1} - \lambda_{i_0}$, $\frac{1}{k+1}\mu_0 = \lambda_{i_0}$

SO $\lambda_{i_k} - \lambda_{i_{k-1}} \ge 0$, $\lambda_{i_{k-1}} - \lambda_{i_{k-2}} \ge 0$, ..., $\lambda_{i_1} - \lambda_{i_0} \ge 0$, $\lambda_{i_0} \ge 0$

- THE FAMILY OF ALL SIMPLEXES OF THE FORM $[b(P_0), \dots, b(P_k)]$ IS A SIMPLICIAL SUBDIVISION OF $[a_0, a_1, \dots, a_k]$
  - THE UNION IS ALL OF S: IF $x \in S$ TAKE A PERMUTATION OF $\{0, \dots, k\}$ WITH $\lambda_{i_k}(x) \ge \dots \ge \lambda_{i_0}(x)$
  - EVERY SIMPLEX IS OF THE FORM $[b(P_{j_0}), \dots, b(P_{j_l})]$ WHERE $i_0, \dots, i_k$ IS A PERMUTATION OF $\{0, \dots, k\}$ THAT GIVES A FULL SEQUENCE $P_0 \supset P_1 \supset \dots \supset P_k$ AND OUR SIMPLEX IS THE FACE OF $[b(P_0) \dots b(P_k)]$ GIVEN BY $\mu_j = 0$ IF $j \notin \{j_0, \dots, j_l\}$