1913 B2 **Mathematics.** — "*Some remarks on the coherence type η.*" By
⟦1⟧ Prof. L. E. J. BROUWER.

⟦2⟧ In order to introduce the notion of a "coherence type" we shall say that a set M is *normally connected*, if to some sequences f of elements of M are adjoined certain elements of M as their "limiting elements", the following conditions being satisfied:

1st. each limiting element of f is at the same time a limiting element of each end segment of f.

⟦3⟧ 2nd. for each limiting element of f a partial sequence of f can be found of which it is the *only* limiting element.

3rd. each limiting element of a partial sequence of f is at the same time a limiting element of f.

4th. if m is the only limiting element of the sequence $\{m_\mu\}$ and

m_μ for μ constant the only limiting element of the sequence $\{m_{\mu\nu}\}$, then each of the latter sequences contains such an end segment $\{m_{\mu\lambda}\}$, that an arbitrary sequence of elements $m_{\mu\lambda}$ for which μ continually increases, possesses m as its only limiting element.

The sets of points of an n-dimensional space form a special case of normally connected sets.

Another special case we get in the following way: In an *n-ply ordered* set [1]) we understand by an *interval* the partial set formed by the elements u satisfying for $\varrho \leq n$ different values of i a relation of the form

$$b_i \overset{i}{<} u \overset{i}{<} c_i \quad \text{or} \quad b_i \overset{i}{<} u \quad \text{or} \quad u \overset{i}{<} c_i \;; \qquad [3a]$$

we further define an element m to be a *limiting element* of a sequence f, if each interval containing m, contains elements of f not identical to m, and the given set to be *everywhere dense*, if none of its intervals reduces to zero. Then the *everywhere dense, countable, n-ply ordered* sets which will be considered more closely in this paper, likewise belong to the class of normally connected sets.

A representation of a normally connected set preserving the limiting element relations, will be called a *continuous representation.*

If of a normally connected set there exists a continuous one-one representation on an other normally connected set, the two sets will be said to possess *the same coherence type.*

One of the simplest coherence types is the type η already introduced by CANTOR [2]). From a proof of CANTOR follows namely:

THEOREM 1. *All countable sets of points lying everywhere dense on the open straight line, possess the same coherence type* η.

The proof is founded on the following construction of a one-one correspondence *preserving the relations of order*, between two sets of points $M = \{m_1, m_2, \ldots\}$ and $R = \{r_1, r_2, \ldots\}$ of the class considered: To r_1 CANTOR makes to correspond the point m_1; to r_2 the point m_{i_2} with the smallest index, having with respect to m_1 the same situation (determined by a relation of order), as r_2 has with respect to r_1; to r_3 the point m_{i_3} with the smallest index, having with respect to m_1 and m_{i_2} the same situation (determined by two relations of order), as r_3 has with respect to r_1 and r_2; and so on. That in this way not only all points of R, but also all points of M have their turn, i.o.w. that if among $m_1, m_{i_2}, \ldots m_{i_\lambda}$ appear $m_1, m_2, \ldots m_\nu$, *but not* $m_{\nu+1}$, there exists a number σ with the property that $m_{\nu+1} = m_{i_{\lambda+\sigma}}$,

[1]) Comp. F. RIESZ, Mathem. Annalen 61, p. 406. [4]

[2]) Mathem. Annalen 46, p. 504. [5]

is evident by choosing for $r_{\lambda+1}$ the point of R with the smallest index, having with respect to $r_1, r_2, \ldots r_\lambda$ the same situation, as $m_{\nu+1}$ has with respect to $m_1, m_{i_2}, \ldots m_{i_\lambda}$. The correspondence constructed in this way, is at the same time *continuous*; for, the limiting point relations depend exclusively on the relations of order, as a point m is then and only then a limiting point of a sequence f, if each interval containing m contains an infinite number of points of f.

The above proof shows at the same time the independence of the coherence type η of the linear continuum. For, after CANTOR it leads also to the following more general result:

THEOREM 2. *All everywhere dense, countable, simply ordered sets possess the coherence type* η.[1])

Theorem 1 may be extended as follows:

THEOREM 3. *If on the open straight line be given two countable, everywhere dense sets of points M and R, a continuous, one-one transformation of the open straight line in itself can be constructed, by which M passes into R.*

In order to define such a transformation, we first by CANTOR's method construct a continuous one-one representation of M on R. Then the order of succession of the points of M is the same as the order of succession of the corresponding points of R. We further make to correspond to each point gm of the straight line *not* belonging to M, the point gr having to the points of R the same relations of order, as gm has to the corresponding points of M. In this way we get a one-one transformation of the straight line in itself, preserving the relations of order. On the grounds indicated in the proof of theorem 1 this transformation must also be a continuous one.

Analogously to theorem 3 is proved:

THEOREM 4. *If within a finite line segment be given two countable, everywhere dense sets of points M and R, a continuous one-one transformation of the line segment, t h e e n d p o i n t s i n c l u d e d, in itself can be constructed, by which M passes into R.*

We shall now treat the question, to what extent the theorems 1, 2, 3, and 4 may be generalized to polydimensional sets of points

[1]) The possibility of a definition founded exclusively on relations of order, shewn by CANTOR not only for the coherence type ν, but likewise for the coherence type ϑ of the complete linear continuum, holds also for the coherence type ζ of the
[6] perfect, punctual sets of points in R_n (comp these Proceedings XII, p. 790). As is easily proved, this coherence type belongs to *all perfect, nowhere dense, simply ordered sets of which the set of intervals is countable* (an "interval" is formed here by each pair of elements between which no further elements lie).

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on one hand, and to multiply ordered sets on the other hand. In the first place the following theorem holds here:

THEOREM 5. *All countable sets of points lying everywhere dense in a cartesian R_n, possess the same coherence type η^n.* [1])

For, to an arbitrary countable set of points, lying everywhere dense in R_n, we can construct a cartesian system of coordinates C_m with the property that no R_{n-1} parallel to a coordinate space contains more than *one* point of the set. If now two such sets, M and R, are given, then in the special case that C_m and C_r are identical, a one-one representation of M on R preserving the n-fold relations of order as determined by $C_m = C_r$, can be constructed by CANTOR's method cited above, only modified in as far as the "situation" of the points with respect to each other is determined here not by simple, but by n-fold relations of order. As on the grounds indicated in the proof of theorem 1 this representation must also be a continuous one, theorem 5 has been established in the special case that C_m and C_r are identical. From this the general case of the theorem ensues immediately.

If on the other hand we have an arbitrary *everywhere dense, countable, n-ply ordered* set Z, then its n simple *projections* [2]), being everywhere dense, countable, and simply ordered, admit of one-one representations preserving the relations of order, on n countable sets of points lying everywhere dense on the n axes of a cartesian system of coordinates successively; these n representations determine together a one-one representation preserving the relations of order, thus a continuous one-one representation of Z on a countable set of points, everywhere dense in R_n. From this we conclude on account of theorem 5:

THEOREM 6. *All everywhere dense, countable, n-ply ordered sets possess the coherence type η^n.*

As the n-dimensional analogon of theorem 3 the following extension of theorem 5 holds:

THEOREM 7. *If in a cartesian R_n be given two countable, everywhere dense sets of points M and R, a continuous one-one transformation of R_n in itself can be constructed, by which M passes into R.* [8]

In the special case that C_m and C_r are identical, we can namely first construct a continuous one-one correspondence between M and R in the manner indicated in the proof of theorem 5, and then make to correspond to each point gm *not* belonging to M, the point gr having to the points of R the same (n-fold) relations of order, as gm has

[1]) This theorem and its proof have been communicated to me by Prof. BOREL. [7]

[2]) Comp. F. RIESZ, l.c. p. 409. [3]

to the corresponding points of M. In this way we get a one-one transformation of R_n in itself *preserving the relations of order as determined by* $C_m = C_r$. As on the grounds indicated in the proof of theorem 1 this transformation is also a continuous one, theorem 7 has been established in the special case that C_m and C_r are identical. From this the general case of the theorem ensues immediately.

The n-dimensional extension of theorem 4 runs as follows:

[9] THEOREM 8. *If within an n-dimensional cube be given two countable, everywhere dense sets of points M and R, a continuous one-one transformation of the cube, the boundary included, in itself can be constructed, by which M passes into R.*

The proof of this theorem is somewhat more complicated than those of the preceding ones. We choose in R_n such a rectangular system of coordinates that the coordinates $x_1, x_2, \ldots . x_n$ of the cube vertices are all either $+1$ or -1, and for $p = 1, 2, \ldots n$ successively we try to form a continuous transition between the $(n-1)$-dimensional spaces $x_p = -1$ and $x_p = +1$ by means of a onedimensional continuum s_{mp} of plane $(n-1)$-dimensional spaces meeting each other neither in the interior nor on the boundary of the cube, and containing each at most one point of M. In this we succeed as follows: Let $S \equiv \sum_{p=1}^{n} a_p x_p = c$ be a plane $(n-1)$-dimensional space containing *no* straight line parallel to a line r_m joining two points of M, and through each point $(x_1 = x_2 = \ldots . = x_{p-1} = 0, x_p = a, x_{p+1} = x_{p+2} = \ldots . = x_n = 0)$ let us lay an $(n-1)$-dimensional space: $x_p + e(1-a^2) S = a + e a_p a (1-a^2)$; in this way we get a continuous series σ_e of plane $(n-1)$-dimensional spaces, and we can choose a magnitude e_1 with the property that for $e < e_1$ two arbitrary spaces of σ_e meet each other neither in the interior nor on the boundary of the cube. As further an $(n-1)$-dimensional space belongs to *at most one* σ_e, thus a line r_m is contained in an $(n-1)$-dimensional space belonging to σ_e for *at most one* value of e, and the lines r_m exist in countable number only, it is possible to choose a suitable value for $e < e_1$ with the property that no space of σ_e contains a line r_m, i.o.w. that σ_e satisfies the conditions imposed to s_{mp}

If for each value of p we choose out of s_{mp} an arbitrary space, then these n spaces possess one single point, lying in the interior of the cube, in common. For, by projecting an arbitrary space of s_{m1} together with the sections determined in it by $s_{m2}, s_{m3}, \ldots . s_{mn}$, into the space $x_1 = 0$, we reduce this property of the n-dimensional cube to the analogous property of the $(n-1)$-dimensional cube. So if we introduce as the coordinate x_{mp} of an arbitrary point H lying in the interior or on the boundary of the cube, the value of x_p in that point of the X_p-axis which lies with H in one and the same space of s_{mp}, then to each system of values > -1 and ≤ 1 for $x_{m1}, x_{m2}, \ldots x_{mn}$ corresponds one and only one point of the interior or of the boundary of the cube, which point is a biuniform, continuous function of $x_{m1}, x_{m2}, \ldots x_{mn}$. I.o.w. the transformation $\{x'_p = x_{mp}\}$, to be represented by T_m, is a continuous one-one transformation of the cube with its boundary in itself, by which M passes into a countable, everywhere dense set of points M_1 *of which no* $(n-1)$-*dimensional space parallel to a coordinate space contains more than one point.*

In the same way we can define a continuous one-one transformation T_r of the cube with its boundary in itself, by which R passes into a countable, everywhere dense set of points R_1 *of which no* $(n-1)$-*dimensional space parallel to a coordinate space contains more than one point.*

Further after the proof of theorem 7 a continuous one-one transformation T of the cube with its boundary in itself exists, by which M_1 passes into R_1, so that the transformation

$$T_r^{-1} \,.\, T \,.\, T_m$$

possesses the properties required by theorem 8.

We now come to a property which at first sight seems to clash with the conception of dimension:

THEOREM 9. *The coherence types* η^n *and* η *are identical.*

To prove this property, in an n-dimensional cube for which the rectangular coordinates of the vertices are all either 0 or 1, we consider the set M_n of coherence type η^n consisting of those points whose coordinates when developed into a series of negative powers of 3, from a certain moment produce exclusively the number 1, and together with this we consider the set M of coherence type η consisting of those real numbers between 0 and 1 which when developed into a series of negative powers of 3^n, from a certain moment produce exclusively the number $\frac{3^n-1}{2}$. The continuous PEANO representation [1]) of the real numbers between 0 and 1 on the n-dimensional cube with edge 1, then determines a *continuous one-one representation of* M *on* M_n establishing the exactness of theorem 9.

That in reality theorem 9 *does not* clash with the conception of dimension, is elucidated by the remark that *not every continuous one-one correspondence between two countable sets of points M and R,*

[1]) Comp. Math. Annalen 36, p. 59, and SCHOENFLIES, Bericht über die Mengenlehre I, p. 125. [[10]]

lying everywhere dense in R_n, admits of an extension to a continuous one-one transformation of R_n in itself. If e.g. the set of the rational points of the open straight line is submitted to the continuous one-one transformation $x' = \frac{1}{\pi - x}$, this transformation *does not* admit of an extension to a continuous one-one transformation of the open straight line in itself.

A more characteristic example, presenting the property moreover that in no partial region an extension is possible, we get as follows: Let t_1 denote the set of those real numbers between 0 and 1 of which the development in the nonal system from a certain moment produces exclusively the digit 4, t_2 the set of the finite ternal fractions between 0 and 1. Let T denote a continuous one-one transformation of the set of the real numbers between 0 and 1 in itself, by which t_1 passes into $t_1 + t_2$, thus a part t_3 of t_1 into t_1, and a part t_4 of t_1 into t_2. By a PEANO representation T_1 the sets t_1, t_2, t_3, t_4 successively pass into countable sets of points s_1, s_2, s_3, s_4, lying everywhere dense within a square with side unity, and, so far as are concerned, s_1, s_3, and s_4, containing no points of the boundary of this square. The continuous one-one representation T of t_3 on t_1 now determines a *continuous one-one representation* $T_2 = T_1 T T_1^{-1}$ *of* s_3 *on* s_1, *not capable of an extension to a continuous one-one representation of the interior of the square in itself.* For, if such an extension would exist, it would be, for each set of points in the interior of the square, the only possible continuous extension of T_2. For s_1, however, $T_1 T T_1^{-1}$ furnishes itself such a continuous extension, which we know to be *not* a one-one representation.

The conception of dimension can now be saved, at least for the everywhere dense, countable *sets of points*, by replacing the notion of *coherence type* by the notion of *geometric type*[1]). Two sets of points will namely be said to possess the same *geometric type*, if a *uniformly continuous* one-one correpondence exists between them. And it is for uniformly continuous representations that the following property holds:

THEOREM 10. *Every uniformly continuous one-one correspondence between two countable sets of points M and R, lying everywhere dense in an n-dimensional cube, admits of an extension to a continuous one-one transformation of the cube with its boundary in itself.*

[6] [1]) For closed sets the two notions are equivalent. For these they were introduced formerly under the name of *geometric type of order*, these Proceedings XII, p. 786.

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For, on account of the uniform continuity of the correspondence between M and R, to a sequence of points of M possessing only one limiting point, a sequence of points of R likewise possessing only one limiting point, must correspond, and reciprocally. On this ground the given correspondence already admits of an extension to a one-one transformation of the cube with its boundary in itself of which we have still to prove the continuity in the property that a sequence $\{g_{m\nu}\}$ of limiting points of M converging to a single limiting point $g_{m\omega}$, the sequence $\{g_{r\nu}\}$ of the corresponding limiting points of R converges likewise to a single limiting point. For this purpose we adjoin to each point $g_{m\nu}$ a point m_ν of M possessing a distance $< \varepsilon_\nu$ from $g_{m\nu}$, the distance between $g_{r\nu}$ and the point r_ν corresponding to m_ν likewise being $< \varepsilon_\nu$, and for ν indefinitely increasing we make ε_ν to converge to zero. Thus $\{m_\nu\}$ converging exclusively to $g_{m\omega}$, $\{r_\nu\}$ likewise possesses a single limiting point $g_{r\omega}$, and also $\{g_{r\nu}\}$ must converge exclusively to $g_{r\omega}$.

On account of the invariance of the number of dimensions [1]) we can enunciate as a corollary of theorem 10:

THEOREM 11. *For $m < n$ the geometric types η^m and η^n are different.*

As, however, for normally connected sets in general the notion of uniform continuity is senseless, the *indeterminateness of the number of dimensions of everywhere dense, countable, multiply ordered sets*, as expressed in theorem 9, must be considered as irreparable.

[1]) Comp. Math. Annalen 70, p. 161.

NOTES

[[1]] The Dutch version was communicated at the meeting of 25 April 1913. The English version appeared in the Proceedings of the meeting of 25 April 1913.
[[2]] The strange looking term 'coherence' is probably derived from G. Cantor's 'Cohärenz' (= set of accumulation points).
[[3]] Brouwer's copy of the Dutch text indicates the changes (translated):

2nd. for each limiting element of f a partial sequence can be found every partial sequence of which has f as its *only* limiting element

4th. if m is the only limiting element of all partial sequences of the sequence $\{m_\mu\}$ and $m_{\mu\nu}$ for μ constant the only limiting element of all partial sequences of the sequence $\{m_{\mu\nu}\}$ then $\cdots$

[[3a]] Added in Brouwer's copy of the Dutch text: $(b_i \overset{i}{<} c_i)$ before the first 'or'.
[[4]] F. Riesz 1905.
[[5]] G. Cantor 1895.
[[6]] Brouwer 1910 B2.
[[7]] This was probably an oral communication, at the International Congress of Mathematicians in Cambridge in August 1912 (see Y58). The theorem itself is superseded by Brouwer's theorem 9. Brouwer's formulations seem to indicate that Borel did not remark that his method allowed him to prove the much stronger theorem 7. Brouwer's papers contain a correspondence with E. Borel, of a later date and related to theorem 8, which he had discussed with Borel at Cambridge, and for which Borel gave an incorrect proof. See 1916 A, 1916 B, Y57, Y58.
[[8]] M. Fréchet 1910, p. 159 had enunciated a theorem which essentially coincides with theorem 7, but the proof was wrong as noticed by P. Urysohn 1925, p. 83–89. Clearly neither Borel nor Brouwer knew about Fréchet's statement nor Urysohn about Brouwer's paper.

M. Fréchet 1928, p. 49 reacted to Urysohn's remark; he had noticed his mistake earlier and found a way of correcting it. M. Fréchet 1928, p. 49 said: 'Dans l'intervalle, d'ailleurs, M. Brouwer avait publié en 1913 une communication de M. Borel, énonçant et démontrant le même théorème.' This is not correct; in any case it does not follow from Brouwer's text I interpret to mean that Borel gave Brouwer a proof of theorem 5 and that Brouwer noticed that the method even allows one to prove theorem 7. In this connection M. Fréchet also mentioned Borel 1922, but not Borel 1913 nor Brouwer's criticism (see [[9]] and 1916 B [[2]]).
[[9]] This theorem is related to Borel 1913 (see Brouwer 1916 A).
[[10]] G. Peano 1890 A, A. Schoenflies 1900.
[[11]] Brouwer 1911 C.

L. E. J. Brouwer. Eenige opmerkingen over het samenhangstype η. Amst. Ak. Versl. **21**, 1412-1419. ⟦1⟧

In der Terminologie des Verf. besitzen zwei Mengen denselben „Zusammenhangstypus", wenn sie sich eineindeutig und stetig aufeinander abbilden lassen, und denselben „geometrischen Typus", wenn diese Abbildung überdies gleichmäßig stetig ist. Es wird gezeigt, daß alle mehrfach geordneten, abzählbaren und überall dichten Mengen denselben Zusammenhangstypus besitzen, nämlich den Typus η der rationalen Zahlen. Weil bei diesen Mengen vom Begriff des geometrischen Typus keine Rede sein kann, so kommt für sie die Invarianz der Dimensionenzahl in Fortfall. Sodann beweist der Verf. folgendes geometrische Theorem: „Wenn im Innern eines n-dimensionalen Kubus zwei abzählbare, überall dichte Punktmengen gegeben sind, so lassen sie sich ineinander überführen mittels einer eineindeutigen und stetigen Transformation des Kubus einschließlich der Grenze in sich." Brw.

NOTE

⟦1⟧ This is an author's review of 1913 B1. See also 1913 B2, 1916 B, Y57. Brouwer's papers contain a manuscript of this review. Deviations are probably due to the editor of *Jahrbuch.*