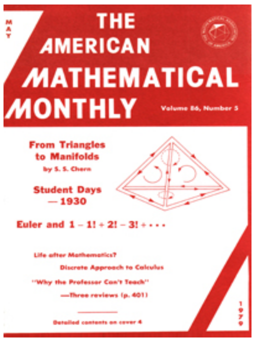

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Euler Subdues a very Obstreperous Series

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would see how little our activities are related to the real needs of society." Fifteen minutes later, he outlined a proof that every sufficiently large integer can be written as a sum, not of 1140 tenth powers (the best previous result), but of 1046 tenth powers.

Acknowledgment. R. P. Boas has acted as my Maxwell Perkins. Some of the anecdotes are his. The errors are all mine.

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EULER SUBDUES A VERY OBSTREPEROUS SERIES

E. J. BARBEAU

The task of evaluating the infinite series $1-1!+2!-3!+\cdots$ caused Euler to clarify his ideas on the meaning of assigning a sum to a series, even one which, in modern eyes, is divergent. In this article, we summarize these ideas and outline four ingenious approaches of Euler to evaluate

The author received his Ph.D. from the University of Newcastle-upon-Tyne under the direction of F. F. Bonsall. He has taught at the University of Western Ontario (London) and since 1967 at the University of Toronto; he spent a year at Yale on a NATO grant. He is interested in the history of mathematics (Euler, in particular), number theory, differential equations, and mathematical education in the schools.—*Editors* the above series. The consistency of these approaches is discussed, with reference to summability methods, extrapolation, continued fractions, and infinite differential operators.

1. Assigning a value to a divergent series. In the late seventeenth and early eighteenth centuries, mathematicians were busily developing what promised to be a significant body of powerful techniques consequent to the creation of the calculus. However, there was considerable uncertainty about the best formulation of the underlying concepts. Probably no better example of this can be found than in the discussion of the meaning of the sum or value of an infinite series. Since sums of monomials can be easily differentiated and integrated, the discovery by Newton and his contemporaries that a great many functions could be developed as power series meant that calculus had quite wide applicability. Consequently, the question of attaching a sum to a series attracted much interest and controversy.

Although mathematicians of this period were aware intuitively that, for some series, the sum could be regarded as the limit of the partial sums, in their view this did not adequately cover the matter. Even when this limit did not exist, many series nevertheless seemed to possess a natural value. Their attitude was influenced in part by their notion of a function as an analytic expression defined over the widest possible domain, including complex numbers and quantities infinitely great or small. Not being in possession of pathological counterexamples, they considered that two analytic expressions agreeing on a continuous set must agree everywhere. Thus, for example, if $(1+x)^c$ is synonymous with its binomial expansion for $|x|<1$, then $(1+x)^c$ must be the value of that expansion for all x, except possibly for obvious singularities.

These opinions were buttressed by experience. It was generally found that, where there were several ways of determining the value of an infinite series, they gave the same result. Moreover, in computations, the practice of interchanging an infinite series with its value did not appear to cause trouble.

Euler's paper, "De seriebus divergentibus" [12], published in 1760, illuminates this spirit well. It can be split into two parts. The first subtly treats the question of assigning a value to a series. The second is devoted to evaluating "Wallis' hypergeometric series"

$$1-1!+2!-3!+4!-5!+\cdots.$$

Here we have a somewhat different approach to mathematical acceptability than that of today. Euler's concern is to put his result beyond all reasonable doubt, and this he does by arriving at it by a number of routes. It is consistency, as much as logical argument, which puts its stamp of approval on the mathematics. (See [15] for a wider discussion of this issue.)

Although the modern investigator would quarrel with details of the work of Euler or of his contemporaries, it nevertheless displays a compelling consistency and usually leads to results demonstrably correct according to today's standards of rigour. Consequently, unusual methods of assigning a value to an infinite series have not been disdained during the past century, but rather formalized, studied in detail, compared, and extended. In situations where normal convergence fails, it is possible to find an alternative definition of "sum" which retains many of the properties associated with the usual concept (and, indeed, agrees with it for series convergent in the normal sense) and which will assign to a given infinite series the value of the function which generates it. This can be done for the binomial expansion and other power series beyond the circle of convergence, as well as for Fourier series, witness Fejér's theorem on the Cesàro-summability of the Fourier expansion of a continuous function. A discussion of summability from the modern point of view can be found elsewhere [5, pp. 5–10], [20], [23].

2. Euler's general outlook. The prospectus to his paper [12] declares Euler's intention "to clarify a concept causing up to now the greatest difficulties." While he would not accept that mathematics is necessarily free of controversy, he is confident that mathematical disputes, unlike those in other areas, can be completely resolved once the evidence has been thoroughly weighed. So it is with assigning values to infinite series. Infinite series can be divided into four categories according as the terms are positive or alternating, bounded or unbounded. Examples of the four groups are

$$\begin{array}{ll} \text{I.} & 1+1+1+1+\cdots \\ \text{II.} & 1-1+1-1+\cdots \\ \text{III.} & 1+2+4+8+\cdots \\ \text{IV.} & 1-2+4-8+\cdots \end{array}$$

Series in group I present no difficulty. Either they converge to a finite sum in the modern sense, or they diverge to the infinite sum, $a/0$. More contentious are the series of group II. Euler bases his discussion on the expansion $1-a+a^2-a^3+\cdots$ of the fraction $1/(1+a)$. While no one would deny that these two expressions agree when $|a|<1$, one might object to assigning the fraction as the sum of the series when $|a|\geqslant 1$ on the grounds that the remainder term $\pm a^{n+1}/(1+a)$ in the equation

$$\frac{1}{1+a}=1-a+a^2-a^3+\cdots\mp a^n\pm\frac{a^{n+1}}{1+a}$$

cannot be neglected. Some of those who support the fraction as sum counter that, for infinite n, the ambiguous sign makes the remainder indeterminate, so that the remainder should be forgotten. In any case, they say, when you sum to infinity, you never reach the place where the remainder has to be inserted. Euler reserves his own position until later.

Those who would assign sums to divergent series appear to be in deep trouble with series in group III. Although it might seem appropriate to assign for these series, as for those of group I, an infinite sum, there occur situations in which the sum indicated by analysis is not only finite but negative. For example, substituting -3 for a in the expansion of $1/(1+a)$ yields the paradoxical equation

$$-\frac{1}{2}=1+3+9+27+81+\cdots.$$

Here one is in the absurd position of adding together positive terms to get a negative sum. Nevertheless, explaining this is a mere challenge to the ingenuity! To resolve the paradox (says Euler), some try to distinguish between two types of negative numbers, those that are less than zero and those that are greater than infinity. An example of the first type is the difference between an integer and its successor: $-1=n-(n+1)$. An example of the second type is $-1=1/-1$, since it fits naturally into the "increasing" sequence

$$\ldots,1/3,1/2,1/1,1/0,1/-1,1/-2,1/-3,\ldots.$$

Euler disapproves of this distinction on the grounds that it "does violence to the certitude of analysis" to have two different concepts of -1. However, he is prepared to accept that "the same quantities which are less than zero can be considered to be greater than infinity."

Series in group IV can sometimes be handled as those in group II, already treated. For example, from the expansion of $1/(1+1)^2$, it is found that $\frac{1}{4}=1-2+3-4+5-6+\cdots$. Euler says very little about this type in general, except to remark that it "is usually burdened with problems of its very own."

Euler affirms that the real justification for assigning a value to a divergent series does not rest in any of the specious arguments given above, but rather in a substitution principle. If an infinite expansion can be replaced in a calculation by the expression which generates it without any ensuing error, then this replacement should be considered valid. One has only to be careful that the rules for doing this are properly investigated. As for the techniques to determine exactly what the value of a given series is, their power can be demonstrated by treating the particularly violent specimen which occupies the rest of the paper.

3. Euler's treatment of Wallis' series. Euler's attribution of the series $1-1!+2!-3!+\cdots$ to Wallis is a mystery. While Wallis had much to say about summing progressions, I have found no reference in his work to this particular series. His interest in the factorial function lay in interpolating its values for non-integral arguments. He discusses this question in the Scholium to Proposition 190 in his *Arithmetica infinitorum* (1655) and, again, in a letter to Leibniz dated January 16, 1699 [**14**, p. 59], where he seeks a formula for $n!$ which makes sense for nonintegral n comparable to the formula $\frac{1}{2}(n^2+n)$ for the sum $1+2+3+\cdots+n$. The adjective "hypergeometric" simply signifies that each term is obtained from its predecessor by multiplying by a factor which varies (presumably in some regular way) from term to term. This is in contrast to a "geometric" progression in which the multiplying factor remains the same. However, with the great interest in the factorial function, it is likely that the problem of summing "Wallis'" series was formulated in the correspondence of the early eighteenth century. Euler himself discussed it in at least two letters to Nicholas Bernoulli [**13**, pp. 538, 543, 546] before publishing his findings in the paper under discussion. The series gets brief mention in the books by Kline [**6**, pp. 451, 1114], Bromwich [**2**, p. 323] and Hardy [**5**, pp. 26–29].

Euler evaluates the series by four different methods. In the first he is content to get a rough numerical approximation by exploiting the fact that the series is alternating. To motivate his approach, let us first consider the sum of an alternating series which converges: $1-1/2+1/3-1/4+1/5-1/6+\cdots=\log 2$. An upper bound for the sum is any partial sum whose last term is positive—for example, $1-1/2+1/3-1/4+1/5=47/60$; a lower bound is any partial sum whose last term is negative—for example, $1-1/2+1/3-1/4=7/12$. Apply the same reasoning to Wallis' series. The partial sums are $1,0,2,-4,20,-100,620,\ldots$. The odd partial sums give upper bounds for the value of the series; the even partial sums give lower bounds. However, because the general term does not tend to zero, we do not obtain a very good estimate. Indeed, all that can be said is that the value lies between 0 and 1. Consequently, we would like to transform the series into an equivalent series which is alternating but which is capable of giving a better estimate.

To see how this might be done, notice that the summing of the alternating series $a_1-a_2+a_3-a_4+a_5-a_6+\cdots$ can be achieved by evaluating the power series

$$a_1x-a_2x^2+a_3x^3-a_4x^4+a_5x^5-a_6x^6+\cdots$$

at $x=1$. We effect a change of variables to produce the required transformation. Introduce y by the equation

$$x=y(1-y)^{-1}=y+y^2+y^3+y^4+\ldots.$$

After substitution and some formal manipulation, the power series becomes

$$a_1y-(\Delta a_1)y^2+(\Delta^2a_1)y^3-\cdots+(-1)^{k-1}(\Delta^{k-1}a_1)y^k+\cdots$$

where Δ is the forward difference operator defined by

$$\Delta^0a_i=a_i,\qquad \Delta^1a_i=\Delta a_i=a_{i+1}-a_i$$

$$\Delta^ka_i=\Delta^{k-1}a_{i+1}-\Delta^{k-1}a_i=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}a_{i+j}$$

for $i\geqslant 1$, $k\geqslant 2$. Since $x=1$ corresponds to $y=\frac{1}{2}$, we can evaluate $a_1-a_2+a_3-a_4+\cdots$ by evaluating the y-series at $y=\frac{1}{2}$:

$$\tfrac{1}{2}a_1-\tfrac{1}{4}(\Delta a_1)+\tfrac{1}{8}(\Delta^2a_1)-\tfrac{1}{16}(\Delta^3a_1)+\cdots.$$

Euler applies this to obtaining the value

$$A\equiv 1-1+2-6+24-120+720-5040+40320-\cdots.$$

First remove the first two terms, $1-1$, which cancel, and divide by 2 to get

$$\frac{A}{2}=1-3+12-60+360-2520+20160-181440$$

2 9 48 300 2160 17640 161280

7 39 252 1860 15480 143640

32 213 1608 13620 128160

181 1395 12012 114540

1214 10617 102528

9403 91911

82508

The rows under the series give, for the absolute values of its terms, differences of the first, second, third, etc., orders, respectively. Applying the transformation, we find that

$$\frac{A}{2}=\frac{1}{2}-\frac{2}{4}+\frac{7}{8}-\frac{32}{16}+\frac{181}{32}-\frac{1214}{64}+\frac{9403}{128}-\frac{82508}{256}+\cdots.$$

Cancelling the first two terms and multiplying by 2 gives

$$A=\frac{7}{4}-\frac{32}{8}+\frac{181}{16}-\frac{1214}{32}+\frac{9403}{64}-\frac{82508}{128}+\cdots.$$

It can be seen that not much progress has been made! However, Euler continues transforming the series, to get at the next turn of the crank,

$$A=\frac{7}{8}-\frac{18}{32}+\frac{81}{128}-\frac{456}{512}+\frac{3123}{2048}-\frac{24894}{8192}+\cdots.$$

Now the second term has smaller magnitude than the first. From the first two partial sums, A must be between $7/8$ and $5/16$. After one more application of the transformation, Euler is prepared to say that A is about 0.580.

The difference operator intervenes also in Euler's second attack on the series. His strategy is to define a sequence whose zeroth term is formally Wallis' series and then to compute this zeroth term numerically. This requires Newton's method of extrapolation, which will be briefly described. For a given sequence, $(a_1,a_2,\ldots,a_n)$, observe that

$$\begin{aligned}a_{m+1}&=a_m+(a_{m+1}-a_m)=a_m+\Delta a_m\equiv(1+\Delta)a_m\\a_{m+2}&=a_{m+1}+\Delta a_{m+1}=(1+\Delta)a_{m+1}=(1+\Delta)(1+\Delta)a_m\\&=a_m+2\Delta a_m+\Delta^2 a_m\end{aligned}$$

and, for any positive integers m and k,

$$\begin{aligned}a_{m+k}&=(1+\Delta)^k a_m\\&\equiv a_m+k\Delta a_m+\frac{k(k-1)}{2}\Delta^2 a_m+\frac{k(k-1)(k-2)}{6}\Delta^3 a_m+\cdots.\end{aligned}\tag{3.1}$$

For k other than a nonnegative integer, the right side of (3.1) still makes sense, so that (3.1) can be used to represent "terms" of the sequence corresponding to indices other than natural numbers.

Euler considers the sequence (P_n) whose terms are given by $P_1=1$, $P_2=2$, $P_3=5$, $P_4=16$, $P_5=65$, and, generally,

$$P_{n+1}=nP_n+1\quad\text{for } n=2,3,4,\ldots,.\tag{3.2}$$

From the fact that $\Delta^i P_1=i!$ $(i=0,1,2,3,\ldots,)$, the formula (3.1) with $m=1$, $k=n-1$, yields a formula for P_n:

$$P_n=(1+\Delta)^{n-1}P_1=P_1+(n-1)\Delta P_1+\binom{n-1}{2}\Delta^2P_1+\binom{n-1}{3}\Delta^3P_1+\cdots$$
$$=1+(n-1)+(n-1)(n-2)+(n-1)(n-2)(n-3)+\cdots.$$

Further, substituting 0 for n gives

$$P_0=1-1!+2!-3!+4!-\cdots.$$

How can a numerical value for P_0 be found?

Euler next applies (3.1) with $m=1$, $k=-1$ to the sequences whose general terms are $a_n=1/P_n$ and $a_n=\log_{10}P_n$. In the first case, the zeroth term is found to be

$$1-(-1/2)+(1/5)-(-3/80)+(-36/1040)-(11271/220376)+\cdots$$
$$=1+0.5+0.2+0.0375-0.0364154-0.0511444+\cdots$$
$$=1.651740 \text{ (Euler's figure).}$$

Taking 1.651740 as $1/P_0$, we have that $P_0=0.60542$. Analysis of the second sequence, $(\log_{10}P_n)$, corroborates this determination of P_0 quite well. The zeroth term of the sequence is

$$0-0.3010300+0.0969100-0.0103000-0.0128666-0.0053006+\cdots$$

and this Euler, using the transformation procedure of his first method, computes as $\bar{1}.7779089$. Thereupon, $P_0=0.59966$.

This method raises two interesting questions. First, are the series obtained for $1/P_0$ and $\log_{10}P_0$ actually convergent? Second, if the terms of one sequence are a certain function of the corresponding terms of another, how reasonable is it to expect that the functional relation will persist to the extrapolated terms as well? This does not always happen; if, for positive integers n, $a_n=n$, $b_n=f(a_n)$ with $f(z)=\sin\pi z/(\pi z)$, $f(0)=1$, then Newton's extrapolation procedure yields $a_0=b_0=0$; but $f(a_0)=1$. One suspects that it is not enough for f to be analytic but that it should have less than exponential growth at infinity as well.

The last two approaches of Euler hinge on finding a closed expression for a power series in x, which, for $x=1$, produces Wallis' series. In the third method, he observes that the power series

$$s(x)=x-1x^2+2x^3-6x^4+24x^5-120x^6+\cdots \tag{3.3}$$

formally satisfies the differential equation

$$s'+(s/x^2)=1/x. \tag{3.4}$$

This first order equation can be solved in the usual way; the solution which vanishes for $x=0$ is

$$s(x)=e^{1/x}\int_0^x\frac{e^{-1/t}}{t}\,dt. \tag{3.5}$$

Using the substitution $v=\exp(1-1/t)$, $t=1/(1-\log v)$, $dt/t=dv/v(1-\log v)$, Euler transforms (3.5) to

$$s(x)=e^{(1/x-1)}\int_0^{e^{1-1/x}}\frac{dv}{1-\log v}. \tag{3.6}$$

For future reference, we record here that, making the substitution $s=1/t-1/x$, the integral can be rendered

$$s(x)=\int_0^\infty\frac{xe^{-s}}{1+xs}\,ds. \tag{3.7}$$

These three integrals yield the following alternative forms for the value of Wallis' series:

$$s(1)=e\int_0^1\frac{e^{-1/t}}{t}\,dt=\int_0^1\frac{dv}{1-\log v}=\int_0^\infty\frac{e^{-s}}{1+s}\,ds, \tag{3.8}$$

of which Euler computes the approximate values of the first and second by the trapezoidal rule.

Euler checks that the second integral of (3.8) ought to give Wallis' series by substituting $y=1$ into the expansion (obtained by integrating by parts),

$$\int_0^y \frac{dv}{1-\log v} = \frac{y}{1-\log y} - \frac{1\cdot y}{(1-\log y)^2} + \frac{1\cdot 2\cdot y}{(1-\log y)^3} - \frac{1\cdot 2\cdot 3\cdot y}{(1-\log y)^4} + \cdots . \tag{3.9}$$

This integral allows for an alternative computation of the value of Wallis' series which Euler does not mention in the paper but which he confides in a letter to Bernoulli [13, p. 546]. The left side of (3.9) is expanded in ascending powers of $(1-v)$, and the series is integrated term by term. Upon substitution of 1 for y, there results

$$1-1+2-6+24-\cdots = 1-1/2+1/6-1/12+1/30 -7/360+19/2520-3/560+\cdots . \tag{3.10}$$

Euler's fourth approach is to obtain a "continued fraction" expansion for the power series

$$u(x)=s(x)/x=1-x+2x^2-6x^3+24x^4-120x^5+\cdots . \tag{3.11}$$

This has the form $1/(1+B)$ where

$$B=\frac{x-2x^2+6x^3-\cdots}{1-x+2x^2-6x^3+\cdots}.$$

In turn, B can be put in the form $x/(1+C)$, with

$$C=\frac{x-4x^2+18x^3-96x^4+\cdots}{1-2x+6x^2-\cdots}.$$

So far, we have found that

$$u(x)=\cfrac{1}{1+\cfrac{x}{1+C}},$$

which, for short, we denote by $u(x)=1/1+x/1+C$, with the convention that each slash incorporates everything which follows. This process is continued: C is written as $x/(1+D)$, D as $2x/(1+E)$, and so on. Carrying on indefinitely, Euler finds

$$u(x)=1/1+x/1+x/1+2x/1+2x/1+3x/1+3x/1+4x/1+4x/\cdots . \tag{3.12}$$

The value of $u(x)$ can be approximated by the convergents obtained by stopping the continued fraction (3.12) at any slash. These are $p_1(x)/q_1(x)=1/1+x$; $p_2(x)/q_2(x)=1/1+x/1+x$; $p_3(x)/q_3(x)=1/1+x/1+x/1+2x$. The nth convergent is $p_n(x)/q_n(x)$, where, for small values of n, $p_n(x)$ and $q_n(x)$ are given in the following table:

n	$p_n(x)$	$q_n(x)$
1	1	$1+x$
2	$1+x$	$1+2x$
3	$1+3x$	$1+4x+2x^2$
4	$1+5x+2x^2$	$1+6x+6x^2$
5	$1+8x+11x^2$	$1+9x+18x^2+6x^3$
6	$1+11x+26x^2+6x^3$	$1+12x+36x^2+24x^3$
7	$1+15x+58x^2+50x^3$	$1+16x+72x^2+96x^3+24x^4$.

In general, for $n \geqslant 1$, these relations hold:

$$\begin{aligned} p_{2n+1}(x)&=p_{2n}(x)+(n+1)xp_{2n-1}(x) \qquad & p_{2n+2}(x)&=p_{2n+1}(x)+(n+1)xp_{2n}(x)\\ q_{2n+1}(x)&=q_{2n}(x)+(n+1)xq_{2n-1}(x) \qquad & q_{2n+2}(x)&=q_{2n+1}(x)+(n+1)xq_{2n}(x). \end{aligned} \tag{3.13}$$

The sum of Wallis' series ought to be $u(1)$. For $x=1$, the convergents of the continued fraction (3.12) are 1/2, 2/3, 4/7, 8/13, 20/34, 44/73, 124/209, 300/501,... forming an apparently convergent sequence.

Euler has another, somewhat curious, way of evaluating the continued fraction at $x=1$. He writes

$$A=1/1+1/1+1/1+2/1+2/1+3/1+3/1+4/1+4/1+5/\cdots/1+10/1+10/1+p$$

where $p=11/1+11/1+12/\cdots/1+15/1+15/1+q$,

where $q=16/1+16/1+17/\cdots/1+20/1+20/1+r$, and

where $r=21/1+21/1+22/1+22/1+23/1+23/\ldots$.

From this

$$A=\frac{491459820+139931620p}{824073141+234662231p}$$

$$p=\frac{2381951+649286q}{887640+187440q}$$

$$q=\frac{11437136+2924816r}{3697925+643025r}.$$

The calculation depends on determining r. If, in the definition of r, we replace the numbers 22,23,24,25,... all by 21, we obtain the approximate equation $r=21/(1+r)$, which is satisfied by

$$r=\tfrac{1}{2}(\sqrt{85}-1)=4.10977\ldots. \tag{3.14}$$

Euler has a second way of finding r. We have that

$$r=21/1+21/1+s=(21+21s)/(22+s),$$

where

$$s=22/1+22/1+t=(22+22t)/(23+t) \quad \text{and} \quad t=23/1+23/1+24/1+24/\cdots.$$

Euler assumes that r, s, and t are in arithmetic progression, so that $r+t=2s$. Since $t=(23s-22)/(22-s)$, he finds that

$$r+t=(2s^2+925s-22)/(484-s^2)=2s,$$

whence $2s^3+2s^2-43s-22=0$. This is solved by an approximate method (Newton's) to obtain $s=4.423$, from which $r=4.31$, $q=3.71645446$, $p=3.0266600163$, $A=0.5963473621372$ (Euler's accuracy). Euler notes that close rational approximations can be obtained from the convergents of the simple continued fraction expansion of A,

$$1/1+1/1+1/2+1/10+1/1+1/1+1/4+1/2+1/2+1/13+1/4+\cdots.$$

Euler's ingenuity has brought forth a variety of ways of handling the seemingly impossible problem of attaching a value to Wallis' series. If these all lead to the same numerical result, then it will reinforce the conclusion that Wallis' series has a natural value and that, within computational error, we have found it. Let us make the test. Euler's first method is crude, but does give the value 0.580. His second gives the values 0.60542 (from $1/P_0$) and 0.59966 (from $\log P_0$). The trapezoidal rule for approximate integration with ten subintervals gives, respectively, 0.59637255 and 0.58734359 for the first and second integrals of (3.8). Formula (3.10) gives about 0.59940472. Using the convergents 124/209 and 300/501 of the expansion (3.12) for $u(1)$ puts the value between 0.5933 and 0.5988. Using (3.14) for r gives $A=0.59634738$, and using $r=4.31$ gives 0.596347362. It can be fairly concluded that these results are consistent. Whatever differences arise seem to reflect the accuracy or efficiency of the method. What should the answer be? Hardy [5, p. 26], by computing (3.7), obtains

$$1-1!+2!-3!+\cdots=-e\left(\gamma-1+\frac{1}{2.2!}-\frac{1}{3.3!}+\cdots\right)$$

where $\gamma=0.5772156649015 33\ldots$ is Euler's constant. The value obtained is about 0.59635. In the remainder of this article, we will explore the compatibility of these methods in more detail.

4. Remarks on Euler's evaluations. Euler's definition of the summation of $a_1-a_2+a_3-a_4+\cdots$ in terms of the summation of $\frac{1}{2}a_1-\frac{1}{4}(\Delta a_1)+(1/8)(\Delta^2 a_1)-\cdots$ is the progenitor of the (E,q) summability method based on the transformation $x=y(1-qy)^{-1}$ [5, Ch. 8]. In his discussion of Wallis' series [5, pp. 28, 196], Hardy points out that no application of Euler's method will convert the series into a convergent one, although by making sufficiently many transformations we can make the error introduced very small by stopping the resulting series at a suitable point. By making an adjustment to this method, Hardy shows how to obtain a value for Wallis' series, $\Sigma_n(-1)^n n!$, which agrees with Euler's third method. Since,

$$n!=\int_0^2 e^{-t}t^n\,dt+\int_2^5 e^{-t}t^n\,dt+\cdots+\int_{2^p+2^{p-1}-1}^{2^{p+1}+2^p-1}e^{-t}t^n\,dt+\cdots,$$

we write formally

$$\sum_n(-1)^n n!=\sum_n\int_0^2(-1)^n e^{-t}t^n\,dt+\sum_n\int_2^5(-1)^n e^{-t}t^n\,dt+\cdots+\sum_n\int_{2^p+2^{p-1}-1}^{2^{p+1}+2^p-1}(-1)^n e^{-t}t^n\,dt+\cdots.$$

Each of the series on the right side is (E,q)-summable for q sufficiently large. However, the size of q required to evaluate the series becomes arbitrarily large with p. Putting in the values obtainable in this way, and adding, we obtain for the right side

$$\int_0^2\frac{e^{-t}}{1+t}\,dt+\int_2^5\frac{e^{-t}}{1+t}\,dt\cdots=\int_0^\infty\frac{e^{-t}}{1+t}\,dt, \tag{4.1}$$

which is the third integral of (3.8).

Euler's third method amounts to assigning the value (3.7) to the power series (3.3), or equivalently the integral $\int_0^\infty e^{-s}(1+xs)^{-1}ds$ to (3.11). His judgment can be vindicated in a number of ways. For example, expanding the integrand we find that

$$\int_0^\infty\frac{e^{-s}}{1+xs}\,ds=1-x+2!x^2-3!x^3+\cdots+(-1)^n n!x^n$$
$$+(-1)^{n+1}x^{n+1}\int_0^\infty\frac{e^{-s}s^{n+1}}{1+xs}\,ds. \tag{4.2}$$

For positive values of x, the integral in the remainder term is dominated by $\int_0^\infty e^{-s}s^{n+1}ds=(n+1)!$, so that the remainder term is of the same sign as and of less magnitude than the term $(-1)^{n+1}(n+1)!x^{n+1}$. While the ratio test reveals that the series (3.11) does indeed diverge for all positive values of x, nevertheless, from (4.2), for each fixed n

$$\int_0^\infty\frac{e^{-s}}{1+xs}\,ds=1-x+2!x^2-3!x^3+\cdots+(-1)^n n!x^n+O(x^{n+1})\quad\text{as } x\to 0.$$

(See [4] for further discussion of this type of behavior.)

Alternatively, an independent evaluation of the power series (3.11) can be made by a modification of Borel's integral method of summability [5, pp. 182, 192], which is quite powerful. Recall that, to sum the series $a_0+a_1+a_2+a_3+\cdots$, we define the function

$$U(s)=\sum_{k=0}^{\infty}\frac{a_k}{k!}s^k. \tag{4.3}$$

Assuming an analytic determination of $U(s)$ for $0\leqslant s<\infty$, the Borel or, more properly, (B^*) sum is defined to be

$$\int_0^\infty e^{-s}U(s)\,ds. \tag{4.4}$$

For the series (3.11), $a_k = (-1)^k k! x^k$, so that

$$U(s) = \sum_{k=0}^{\infty} (-1)^k (xs)^k.$$

When $|xs| < 1$, $U(s)$ is equal to $(1+xs)^{-1}$; we take this as an evaluation of $U(s)$ for all nonnegative s. Then the (B^*) sum of (3.11) will be

$$\int_0^{\infty} e^{-s}(1+xs)^{-1}\,ds. \tag{4.5}$$

Another treatment appears in a 1941 paper of Hardy [16]. In effect, he defines the value of $\sum_{n=0}^{\infty}(-1)^n n! x^n$ to be the limit, as $\delta \to 0$, of

$$\sum_{n=0}^{\infty} (-1)^n e^{-\delta\lambda_n} n! x^n \tag{4.6}$$

where $\lambda_0 = \lambda_1 = \lambda_2 = 0$ and $\lambda_n = n \log n \log\log n$ for $n \geq 3$. Through the residue theorem, (4.6) can be expressed as a sum of two contour integrals which tends, as $\delta \to 0$, to the limit

$$\int_C \frac{\Gamma(s+1)}{e^{2\pi i s} - 1}(-x)^s\,ds,$$

for any complex number x not lying on the negative real axis, where C is a suitable contour. Using the integral form of $\Gamma(s+1)$ and effecting an inversion of the order of integration, Hardy obtains from this integral the required quantity (4.5).

The consistency between Euler's third method, using the differential equation for $s(x)$, and his fourth method, using the continued fraction for $u(x)$, was established by both Laguerre [18] and Stieltjes [22]. Both do this by showing that a suitable integral can be developed as a continued fraction. Laguerre treats $\int_x^{\infty} e^{-t}/t\,dt$ which, for $z = 1/x$, $t = 1/s$, becomes $\int_0^z e^{-1/s}/s\,ds$ (cf. (3.5)). On the other hand, Stieltjes substitutes $z = 1/x$ into (3.12) to get a continued fraction of the form

$$1/a_1 z + 1/a_2 + 1/a_3 z + 1/a_4 + 1/a_5 z + \cdots. \tag{4.7}$$

The convergents of this continued fraction are rational functions possessing partial fraction decompositions of the type $\sum M_i/(z + x_i)$ with both M_i and x_i positive. Stieltjes writes the nth convergent as a special kind of integral

$$\int_0^{\infty} \frac{d\phi_n}{z+y}$$

where ϕ_n is an increasing jump function, $\phi_n(0) = 0$ and $\lim_{y\to\infty}\phi_n(y) = 1/a_1$. For this particular case, as n increases, the functions $\phi_n(y)$ tend to a limit $\Phi(y)$, an increasing function determinable by solving the moment problem

$$\int_0^{\infty} t^k\,d\Phi(t) = c_k \qquad (k = 0, 1, 2, \cdots)$$

where the c_k are derivable from the known quantities a_i. Thus, (4.7) can be written as $\int_0^{\infty} d\Phi(t)/(z+t)$ from which (3.12) can be expressed as

$$\int_0^{\infty} \frac{e^{-t}}{1+xt}\,dt. \tag{4.8}$$

However, I will take a more direct approach in connecting the continued fraction for $u(x)$ to its differential equation. Since $u(x) = s(x)/x$ and since s satisfies (3.4), the differential equation for u is

$$x^2 u' + (1+x)u = 1 \tag{4.9}$$

(which can be seen to have the solution (4.8) with $u(0) = 1$). Our task is to show, first, that for

$0<x<\infty$, the sequence $(p_n(x)/q_n(x))$ of convergents of (3.12) converges, and, second, that the limit is a solution of (4.9).

With the help of (3.13), it is straightforward to establish that, for $n \geq 1$:

(1) $p_{2n}, p_{2n+1}, q_{2n-1}, q_{2n}$ are polynomials of degree n whose coefficients are all nonnegative and whose constant terms are equal to 1.

(2) The leading coefficient of both q_{2n} and q_{2n+1} is $(n+1)!$.

(3) $p_n q_{n+2} - p_{n+2} q_n = p_n q_{n+1} - p_{n+1} q_n$.

(4) Let

$$w_n = p_n q_{n+1} - p_{n+1} q_n. \tag{4.10}$$

Then

$$w_{2n} = -(n+1)xw_{2n-1} \quad \text{and} \quad w_{2n+1} = -(n+1)xw_{2n},$$

so that

$$w_{2n} = (n+1)(n!)^2 x^{2n+1} \tag{4.11}$$
$$w_{2n-1} = -(n!)^2 x^{2n}.$$

(5)

$$\frac{p_{2n}}{q_{2n}} - \frac{p_{2n+2}}{q_{2n+2}} = \frac{(n+1)(n!)^2 x^{2n+1}}{q_{2n+2} q_{2n}}$$
$$\frac{p_{2n+1}}{q_{2n+1}} - \frac{p_{2n-1}}{q_{2n-1}} = \frac{(n!)^2 x^{2n}}{q_{2n+1} q_{2n-1}}$$
$$\frac{p_{2n}}{q_{2n}} - \frac{p_{2n+1}}{q_{2n+1}} = \frac{(n+1)(n!)^2 x^{2n+1}}{q_{2n+1} q_{2n}}$$
$$\frac{p_{2n}}{q_{2n}} - \frac{p_{2n-1}}{q_{2n-1}} = \frac{(n!)^2 x^{2n}}{q_{2n} q_{2n-1}}.$$

Because of (1) and (2), for all positive values of x, the right sides of all four equations are positive and of the last two are each less than $1/(n+1)$. Therefore, it follows that, on the positive real axis, the odd convergents increase and the even convergents decrease to a common limit function $u(x)$.

In order to check that the limit of the convergents satisfies (4.9), we examine the expression $x^2u' + (1+x)u - 1$, with u replaced by p_n/q_n. This is a fraction, whose numerator is

$$x^2(p_n' q_n - p_n q_n') + r_n q_n, \tag{4.12}$$

where $r_n = (1+x)p_n - q_n$ is a polynomial satisfying the same recursion relations (3.13) as p_n and q_n.

It is readily conjectured that, for $n = 1, 2, \ldots$ and $q_0 = 1$,

$$q_{2n+1}' = (n+1)^2 q_{2n-1}$$
$$q_{2n}' = n(n+1) q_{2n-2}, \tag{4.13}$$

which an induction argument shows to be indeed true. This suggests that we should examine $p_{2n+1}' - (n+1)^2 p_{2n-1}$ and $p_{2n}' - n(n+1)p_{2n-2}$. Again, an induction argument reveals that, for $n = 1, 2, \ldots$ and $p_0 = 0$,

$$x^2\big(p_{2n+1}' - (n+1)^2 p_{2n-1}\big) + r_{2n+1} = 0$$
$$x^2\big(p_{2n}' - n(n+1)p_{2n-2}\big) + r_{2n} = 0. \tag{4.14}$$

Inserting (4.13) and (4.14) into (4.12), we find that, for every positive integer n,

$$x^2(p_n'q_n - p_nq_n') + (1+x)p_nq_n - q_n^2 = w_n,$$

where w_n is given by (4.10). Therefore,

$$\left(\frac{p_n}{q_n}\right)' + \left(\frac{1+x}{x^2}\right)\left(\frac{p_n}{q_n}\right) - \frac{1}{x^2} = \frac{w_n}{x^2q_n^2} \tag{4.15}$$

for $x > 0$. From the above facts (1), (2), and (4),

$$\frac{w_{2n}}{x^2q_{2n}^2} \leqslant \frac{(n+1)(n!)^2x^{2n+1}}{x^2((n+1)!)^2x^{2n}} = \frac{1}{(n+1)x}$$

so that, as n tends to infinity through even values, the right side of (4.10) converges uniformly to zero on compact subsets of $(0,\infty)$. Since p_n/q_n tends uniformly to $u(x)$ on compact subsets of $(0,\infty)$, from (4.15), $\lim_{n\to\infty}(p_{2n}/q_{2n})'$ exists and must be $u'(x)$ for $x>0$. Therefore, $u' + (1+x)x^{-2}u - x^{-2} = 0$, so that u satisfies (4.9).

Actually, the convergents give an efficient approximation to $u(x)$ in the sense that the series expansion of $p_n(x)/q_n(x)$ reproduces the terms in the series (3.11) up to degree n, which is about twice the degree of the numerator or the denominator. Indeed, $q_n(x)u(x) = p_n(x) +$ (terms of degree exceeding n), so that the convergents are among the Padé approximants of $u(x)$ [**10**, Ch. 20]. The properties of these convergents also give some insight into the dismal failure of (3.11) to converge while at the same time $u(x)$ can be evaluated for all nonnegative real x. Consider the real zeros of the polynomials $q_n(x)$. It is clear that they must all be negative. By inspection, we note that q_1 and q_2 each have a negative zero. Suppose that for $k \geqslant 2$ it has been established that for $i = 1,2,3,\ldots,k$, q_i has at least one negative zero and that, if b_i is the largest real zero of q_i, then $b_{i-1} < b_i$ for $2 \leqslant i \leqslant k$. Thus, if $1 \leqslant i \leqslant j \leqslant k$, $q_i(x) > 0$ for $b_j < x \leqslant 0$. Then, from (3.13), we can deduce that $q_{k+1}(b_k) < 0$, so that q_{k+1} has a zero in the interval $(b_k, 0)$. In particular, b_{k+1} exists and $b_k < b_{k+1} < 0$. Therefore, $\lim b_n$ exists. We now show that the limit is 0. Substituting $x = -1/(n+1)$ into the second and fourth equations of (3.11), we find that $q_{2n+2}(-1/(n+1)) = -q_{2n-1}(-1/(n+1))$. With a little arguing, it follows that $b_{2n+2} > -1/(n+1)$, so that indeed $\lim b_n = 0$. Consequently, the radius of convergence of the power series expansion of $p_n(x)/q_n(x)$ becomes arbitrarily small for n sufficiently large, while at the same time the singularities keep away from the positive axis. It is not surprising that while (3.11) diverges, the convergents p_n/q_n can be used to assign a value to (3.9) for nonnegative x. In the next section, we will take up Euler's second method.

5. Extrapolation and factorial series. Euler's second method of evaluation requires the value of P_0 given that, for each positive integer n, $P_n = 1 + (n-1) + (n-1)(n-2) + (n-1)(n-2)(n-3) + \cdots$. This could be obtained if a natural extension function $h(z)$ were found which is in some sense regular and defined on a region of the complex plane containing the nonnegative integers, and for which $h(n) = P_n$ for positive integers n.

In fact, there is a great deal of latitude in choosing the function $h(z)$, even if we require it to be entire. By the Weierstrass Theorem [**1**, p. 194], there is an entire function $f(z)$ with simple zeros at $1,2,3,\ldots$. The Mittag-Leffler Theorem [**1**, p. 185] then provides a meromorphic function $g(z)$ whose only poles occur at $1,2,3,\ldots$ and whose singular part at $z = n$ is

$$\frac{P_n}{f'(n)(z-n)}.$$

If $h(z) = f(z)g(z)$, then $h(z)$ is entire and $h(n) = P_n$ $(n = 1,2,3,\ldots)$. If this seems too arbitrary, we can also require that the recursion relation (3.2) satisfied by P_n also extends:

$$h(z+1) = zh(z) + 1 \quad \text{for } z \in C. \tag{5.1}$$

Even this will not pin h down. For, as pointed out by Lorne Campbell, if $h(z)$ is one such function, then $c\Gamma(z)\sin 2\pi z + h(z)$ is another for any constant c. We may well ask whether there is a "right" extension for P_n in the sense that the gamma function is the "right" extension of $n!$.

Following Euler, we try to get one by Newtonian extrapolation. Let the sequence $(a_1, a_2, a_3, \ldots)$ be given and let the ith order difference at the first term be u_i:

$$u_i = \Delta^i a_1 \qquad (i = 0, 1, 2, 3, \ldots). \tag{5.2}$$

Newton's extrapolation formula for a function $f(z)$ with $f(n) = a_n$ for positive integral n is

$$\begin{aligned} f(z) &= (1+\Delta)^{z-1} a_1 \\ &= u_0 + u_1(z-1) + \frac{u_2}{2!}(z-1)(z-2) + \cdots \\ &\quad + \frac{u_k}{k!}(z-1)(z-2)(z-3)\cdots(z-k) + \cdots . \end{aligned} \tag{5.3}$$

This is a factorial series of the second kind, whose theory is expounded in great detail by Nörlund [8]. For such a series, there are two real numbers θ_0 and θ_1, both finite or both infinite, which, when both finite, satisfy $0 \leqslant \theta_1 - \theta_0 \leqslant 1$, such that the series converges for $\operatorname{Re} z > \theta_0$ and converges absolutely for $\operatorname{Re} z > \theta_1$. Convergence is uniform on compact subsets of the domain of convergence, so that on such subsets $f(z)$ is analytic. This extrapolation has some agreeable properties. If, for some polynomial p of degree m, $a_n = p(n)$, then $u_k = 0$ for $k \geqslant m+1$ and $f(z) = p(z)$. In particular, the factorial series corresponding to the sequence $(1, 1, 1, \ldots)$ is simply 1. Moreover, suppose that $f(z)$ and $g(z)$ are the factorial series arising from the sequences (a_n) and (b_n). Then clearly the factorial series from (ca_n) is $cf(z)$ and from $(a_n + b_n)$ is $f(z) + g(z)$. Not so clear, but also true, is the fact that the factorial series from the product sequence $(a_n b_n)$ is the product $f(z)g(z)$ expanded in ascending factorial powers $1, (z-1), (z-1)(z-2), (z-1)(z-2)(z-3), \ldots$. From this, it follows that, if $b_n = 1/a_n$, then $f(z)g(z) = 1$. More generally, if ϕ is a rational function for which $b_n = \phi(a_n)$, then $g(z) = \phi \circ f(z)$. In the light of this, it is entirely reasonable that Euler should extend (P_n) to P_0 by Newton's method and should use the inverse sequence $(1/P_n)$ to compute "$1/P_0$." Is his use of the sequence $(\log_{10} P_n)$ equally justifiable?

Unfortunately, for $a_n = P_n$, the series (5.3) converges nowhere except at the positive integers, where it terminates. Therefore we will come to an assessment of the consistency of Euler's second method with his others by summing (5.3) by various natural methods. For the modified method of Borel discussed above, the function $U(s)$ of (4.3) is

$$U(s) = \sum_{k=0}^{\infty} \frac{(z-1)(z-2)\cdots(z-k)}{k!} s^k .$$

For $|s| < 1$, this is equal to $(1+s)^{z-1}$, and this expression can be used to extend $U(s)$ along the positive real axis. Then, from (4.4), we find that

$$h(z) = \int_0^\infty e^{-s}(1+s)^{z-1}\,ds. \tag{5.4}$$

(The value of $(1+s)^{z-1}$ derives from the real logarithm of $(1+s)$.) The integral converges for all complex values of z and $h(z)$, so determined, is entire. Integrating by parts reveals that $h(z)$ satisfies (5.1); if $z = n$, a positive integer, then $h(n) = P_n$. Finally, for $z = 0$, h takes the value (3.8). We can recover the factorial series $1 + (z-1) + (z-1)(z-2)(z-3) + (z-1)(z-2)(z-3) + \cdots$ from (5.4) by expanding the binomial in the integrand and integrating (invalidly) term by term. This relationship between function and factorial series can be made less tenuous. Define the incomplete gamma function

$$\Gamma(z, x) = \int_x^\infty e^{-t} t^{z-1}\,dt = e^{-1} \int_{x-1}^\infty e^{-t}(1+t)^{z-1}\,dt$$

for nonnegative real x and complex z. Then $h(z) = e\Gamma(z, 1)$. Successive integration by parts yields

$$\Gamma(z,x)=e^{-x}x^{z-1}\left\{1+\frac{(z-1)}{x}+\frac{(z-1)(z-2)}{x^2}+\cdots+\frac{(z-1)(z-2)\cdots(z-k+1)}{x^{k-1}}\right\}+R_k(z,x)$$

where $R_k(z,x)=(z-1)(z-2)\cdots(z-k)\int_x^\infty e^{-t}t^{z-k-1}\,dt$. For fixed z, this remainder becomes arbitrarily small for large x. In the language of asymptotic expansions [4, 9], we have

$$e\Gamma(z,x)\sim e^{1-x}x^{z-1}\sum_{k=0}^{\infty}\frac{(z-1)(z-2)\cdots(z-k)}{x^k}\quad\text{as } x\to\infty,$$

the right side of which, for $x=1$, gives the factorial series for $h(z)$.

There are other methods, somewhat ad hoc but nevertheless convincing, of obtaining the same closed expression for $h(z)$. An amusing way begins with the observation that the sum of the first n terms of the series $e=1+1/1!+1/2!+\cdots$ is $P_n/(n-1)!$. This suggests that it might be worthwhile to consider the x-polynomial

$$q(n,x)=1+x+x^2/2!+x^3/3!+\cdots+x^{n-1}/(n-1)!.$$

We have that $q(n,0)=1$, $P_n=\Gamma(n)q(n,1)$, and $q(n,x)$ satisfies

$$\frac{\partial}{\partial x}q(n,x)-q(n,x)=-x^{n-1}/(n-1)!.$$

Define the extension $q(z,x)$ of $q(n,x)$ for z complex (but not a nonpositive integer) as that function which satisfies the differential equation

$$\frac{\partial}{\partial x}q(z,x)-q(z,x)=-x^{z-1}/\Gamma(z)\qquad(x\geqslant 0)$$

subject to the initial condition $q(z,0)=1$. This first order equation has the solution

$$q(z,x)=\frac{e^x}{\Gamma(z)}\int_x^\infty e^{-t}t^{z-1}\,dt.$$

The resulting extension of P_n is thus

$$\Gamma(z)q(z,1)=e\int_1^\infty e^{-t}t^{z-1}\,dt=\int_0^\infty e^{-s}(1+s)^{z-1}\,ds,$$

which agrees with (5.4), when analytically continued to all of C.

The foregoing approach is peculiar to the sequence (P_n). We put forward a final method which is more general and involves infinite differential operators (cf. [3]). Let (a_n) be a given sequence, and, as above, let $u_i=\Delta^i a_1$ $(i=0,1,2,\ldots)$. Define $\chi_k(t)=t^k$ for $k=0,1,2,\ldots$. Observe the similarity between the expansion of $f(z)$ in (5.3) and the binomial expansion

$$(1+t)^{z-1}=\sum_{k=0}^{\infty}\frac{(z-1)(z-2)\cdots(z-k)}{k!}\chi_k(t).$$

We search for a linear functional, L_a (the subscript indicates its dependence on the sequence (a_n)), defined on a linear space which contains all the functions $\chi_k(t)$ as well as $(1+t)^{z-1}$ for complex values of z, for which

$$L_a(\chi_k)=u_k\quad\text{for } k=0,1,2,\ldots.$$

Then we might take as an evaluation of the series (5.3)

$$f(z)=L_a((1+t)^{z-1}).$$

We should choose L_a in such a way that, if for some polynomial p, $a_n=p(n)$, then we should obtain the evaluation $f(z)=p(z)$.

Let us denote differentiation with respect to t by D or by $'$. Observe that $D^k\chi_r(0)=0$ when

$r \neq k$ while $D^r\chi_r(0)=r!$. Consequently, for suitable $w(t)$, define

$$L_a(w(t)) = u_0 w(0) + \frac{u_1}{1!}w'(0) + \frac{u_2}{2!}w''(0) + \cdots + \frac{u_r}{r!}w^{(r)}(0) + \cdots$$

where the series can be evaluated somehow. Three examples will illustrate the process.

EXAMPLE 1. Let p be any polynomial and let $a_n = p(n)$. Then $u_i = \Delta^i p(1)$ and

$$L_a((1+t)^{z-1}) = p(1) + \frac{\Delta p(1)}{1!}(z-1) + \frac{\Delta^2 p(1)}{2!}(z-1)(z-2) + \cdots + 0 + 0 + 0 + \cdots$$

which is the factorial expansion of the polynomial $p(z)$.

EXAMPLE 2. Let $a_n = 1/n$. Then $u_i = (-1)^i/(1+i)$, so that

$$\begin{aligned} L_a(w(t)) &= w(0) - w'(0)/2! + w''(0)/3! - w'''(0)/4! + \cdots \\ &= \left(1 - \frac{D}{2!} + \frac{D^2}{3!} - \frac{D^3}{4!} + \cdots\right)w(0) \\ &= \frac{1}{D}(1-e^{-D})w(0). \end{aligned}$$

By Taylor's Theorem, formally,

$$\begin{aligned} (1-e^{-D})w(t) &= w(t) - w(t) + w'(t) - w''(t)/2! + \cdots \\ &= w(t) - w(t-1). \end{aligned}$$

Thus,

$$\begin{aligned} (1 - D/2! + D^2/3! - D^3/4! + \cdots)w(t) &= D^{-1}(1-e^{-D})w(t) \\ &= D^{-1}(w(t) - w(t-1)) \\ &= \int_{t-1}^{t} w(s)\,ds + C, \end{aligned}$$

where D^{-1} can be interpreted as integration and C is the constant of integration. In the case that $w(t) = \chi_k$,

$$\begin{aligned} &(1 - D/2! + D^2/3! - D^3/4! + \cdots)\chi_k \\ &\quad = t^k - kt^{k-1}/2! + k(k-1)t^{k-2}/3! + \cdots + 0 + 0 + 0 + \cdots \\ &\quad = (k+1)^{-1}\left((k+1)t^k - \binom{k+1}{2}t^{k-1} + \cdots + (-1)^{r-1}\binom{k+1}{r}t^{k-r+1} + \cdots\right) \\ &\quad = \frac{t^{k+1} - (t-1)^{k+1}}{k+1} = \int_{t-1}^{t} s^k\,ds. \end{aligned}$$

Agreement with the previous calculation demands that we take the constant C to be zero. This we do for arbitrary $w(t)$. We are in a position to evaluate $f(z)$ by taking $w(t) = (1+t)^{z-1}$. Since

$$D^{-1}(1-e^{-D})(1+t)^{z-1} = \int_{t-1}^{t}(1+s)^{z-1}\,ds = \frac{(1+t)^z - t^z}{z},$$

we find that $f(z) = 1/z$, as we would wish.

EXAMPLE 3. Let the sequence be (P_n). Since $\Delta^i P = i!$, $L_P(w(t)) = (1 + D + D^2 + \cdots + D^k + \cdots)w(0) = (1-D)^{-1}w(0)$. Let $y(t) = (1-D)^{-1}w(t)$. Then $(1-D)y(t) = w(t)$, i.e., y is a solution of the differential equation $y' - y = -w$. Thus,

$$y(t) = e^t \int_t^{\infty} e^{-s} w(s)\,ds + Ce^t.$$

For the particular case $w(t)=\chi_k(t)$, we find that

$$(1+D+D^2+\cdots+D^k+\cdots)w(0)=k! \quad \text{while } y(0)=k!+C.$$

Again, this indicates that 0 is the appropriate choice for the constant C. Hence, the extension for P_n is

$$L_P((1+t)^{z-1})=\int_0^\infty e^{-s}(1+s)^{z-1}\,ds,$$

which agrees with (5.4).

This means of summation might be compared with that of Hughes [17] for summing a factorial series of the form $\Sigma v_n/z(z+1)(z+2)\cdots(z+n)$, in that integration plays a role similar to that of differentiation here.

The particular form of $h(z)$ suggests that we consider a linear functional defined by integrating over the positive real half-line. Thus, given a sequence (a_n), we might seek out a measure μ for which

$$\int_0^\infty t^k\,d\mu=\Delta^k a_1 \quad \text{for } k=0,1,2,\ldots.$$

This is the Stieltjes moment problem; it is always solvable, the solution not being unique [25]. To get an extension function $f(z)$ with $f(n)=a_n$ for positive integers n, we choose any μ for which $(1+t)^{z-1}$ is integrable, and compute $f(z)=\int_0^\infty(1+t)^{z-1}\,d\mu$. The problem is to find a systematic way of choosing the "right" measure; how could we be led to the choice of $e^{-t}dt$ in the case of (P_n)? Going back to the polynomials does not seem to be of much help here, involving as it does an inversion of the partial Mellin transform

$$p(z)=\int_0^\infty(1+t)^{z-1}\,d\mu_p$$

for the measure μ_p.

We conclude this section with one quick observation on the infinite series (5.3) evaluated at $z=0$. If b_n is the sum of the first n terms, it can be shown that, in terms of the given sequence (a_n),

$$b_n=\sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}a_i.$$

If Q is the infinite matrix implementing the transformation from (a_n) to (b_n), it is not hard to show that Q^2 is the identity matrix. Is this indicative of some deeper structure? Can this be exploited to justify Euler's consideration of $(1/P_n)$ and $(\log_{10}P_n)$ in computing P_0?

6. Closing Remarks. While finding a sum for Wallis' series is hardly of great mathematical significance, there is some fascination attached to the problem. Doubtless, Euler's analysis can be the starting point for a deeper excursion into mathematical interrelationships in a variety of areas—asymptotic expansions, continued fractions, summability, moment problems, factorial series, rational function approximations, infinite differential operators. In this, as in Euler's other investigations, the breadth and ingenuity justify study in something close to the original form by mathematical students.

It could be pointed out that the difficulty of showing that Wallis' series has a value is a result of the field in which we chose to operate. For any prime p, it is clear that Wallis' series converges in the p-adic completion of the rationals, and to an integer, too!

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MATHEMATICAL NOTES

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SUMS OF RECIPROCALS OF INTEGERS MISSING A GIVEN DIGIT

ROBERT BAILLIE

The harmonic series $\Sigma_{n=1}^{\infty} 1/n$ diverges. If we omit those terms for which n has, say, at least one "9" in its base 10 representation, then the remaining series converges [6]. In fact, this result holds for any base $b \geqslant 2$ and any digit m, $0 \leqslant m \leqslant b-1$. (See [4, Theorem 144, p. 120].) Various