2 Key Mathematics Questions Answered After Quarter Century

Proof Concerns Theory of Sets, Widely Used in Teaching Beginners— Work Is Discussed at Seminar

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Two of the most fundamental questions in mathematics today have been answered by Paul J. Cohen, a young Stanford University mathematician.

The two questions had persisted for more than a quarter century. By answering them, Dr. Cohen demonstrated the power of modern mathematics. Ironically, he exposed some of its weakness as well.

The proof, which will be published soon, concerns a branch of mathematics known as the theory of sets, which deals with collections of things. Dr. Cohen's work has meaning, however, for the foundations of all mathematics.

The theory of sets was developed by Georg Cantor, a German, in 1871. Today, it provides part of the basis for the so-called "new mathematics" that is being taught in many elementary and high schools across the country.

The idea behind the use of set theory as a teaching aid in beginning mathematics is that concepts such as those of numbers, counting and arithmetic can be more easily understood by seeing how things can be grouped together and how various collections of things are related to one another than by the rote learning of rules.

Called Research Tool

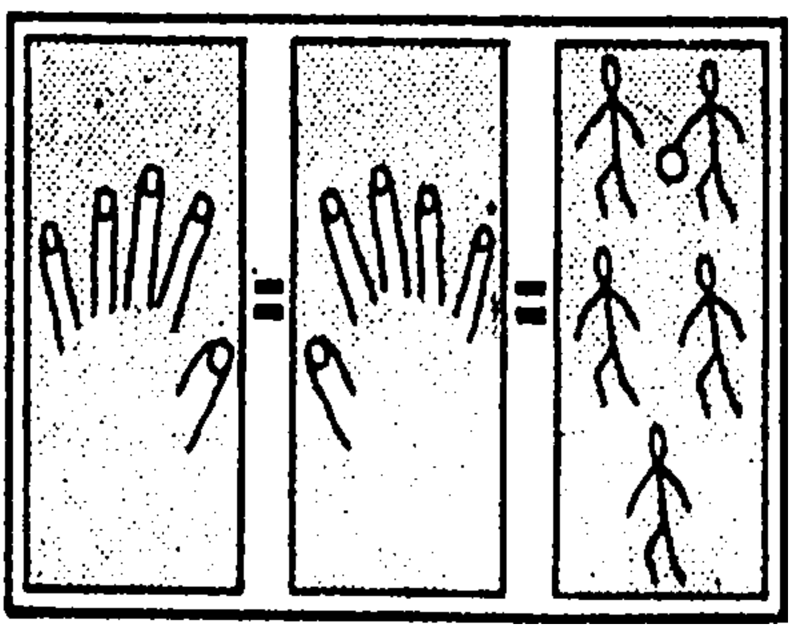
Moreover, set theory serves as a powerful research probe into the foundations of mathematics. In fact, one mathematician observed recently that all of mathematics could be expressed in terms of the theory of sets.

A brief introduction to set theory and the contribution to it made by Dr. Cohen were presented at a seminar on mathematics for science writers recently by Prof. Raymond M. Smullyan of Yeshiva University.

The seminar was held at Columbia University under the auspices of the Society for Industrial and Applied Mathematics and the Martin Company's Research Institute for Advanced Studies. The following is how Professor Smullyan explained things.

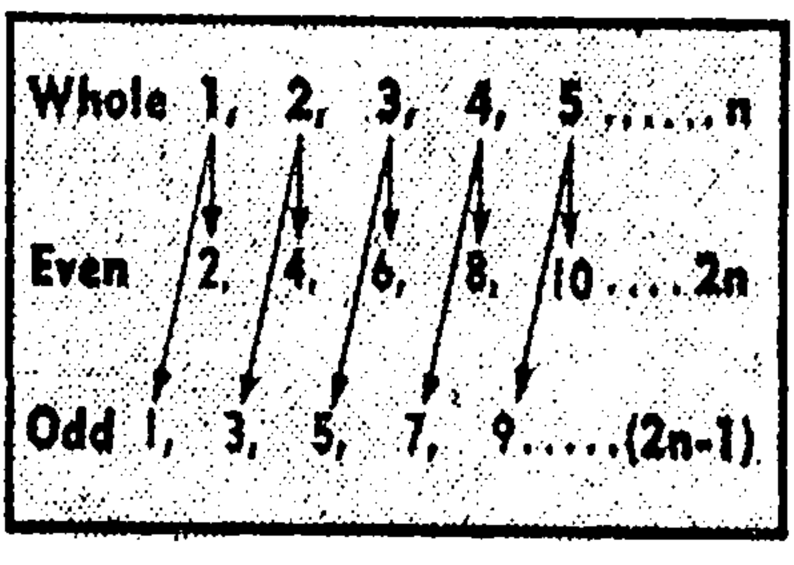
A set is a collection of things —numbers, points on a line, various objects, and so forth. An important characteristic of a set is the number of things it contains. This is called the set's cardinal number. All sets whose elements can be put into one-to-one correspondence with the elements of each other are said to be equivalent and have the same cardinal number.

For example, the set consisting of the fingers on the left hand is equivalent to the set of fingers on the right and to the set of men on a basketball team because the elements of each set can be put into a one-to-one relationship with those of each of the others. Their common cardinal number is five.



Those sets would not be equivalent, of course, to the set of dimes in a dollar or the set of wheels on a car, because the cardinal numbers of those sets are 10 and 4, respectively.

There are infinitely large sets too. The set of all whole numbers—that is, those without fractions (0, 1, 2, 3, etc.)—is an example of an infinite set. It is equivalent to the set of all even numbers because, for every whole number, it is possible to find an even number to correspond. For the same reason, the set of all odd numbers and the set of all pairs of whole numbers is, each, equivalent to the set of all whole numbers.



Cantor gave the name "aleph-null" to the cardinal number of sets of all such finite numbers (0, 1, 2, 3, 4, etc.). It might seem that all infinite sets would have this same cardinal

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number, but Cantor showed otherwise.

He showed, first, that such sets as the set of all real numbers between zero and one (decimals, that is), the set of all points on a line and the collection of all sets of whole numbers have a higher cardinal number than aleph-null. He called this cardinal number "c," for the "power of the continuum."

Cantor showed that the cardinal number, c, is higher than aleph-null by the following sort of proof.

Proof Is Outlined

Make a collection of all conceivable sets of whole numbers such that, arbitrarily, Set 1 might contain all numbers between 10 and 50, Set 2 might have all even numbers between 1 and 25, Set 3 all numbers between 36 and 92, etc.

Next, match each set with each whole number in the set of all whole numbers so that Set 1 goes with "one," Set 2 with "two," etc.

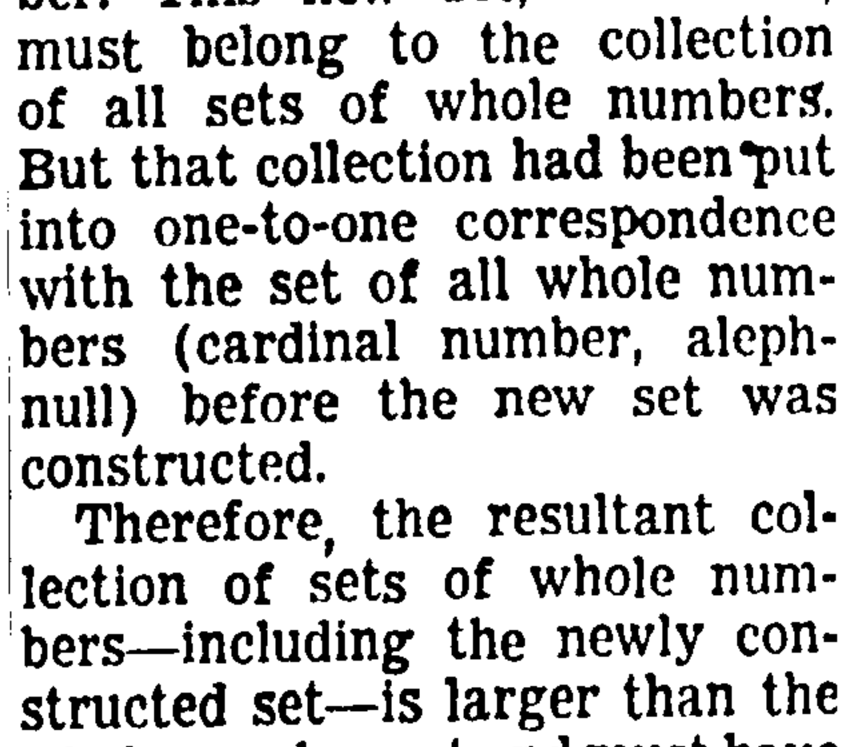
Now, make a new set of whole numbers in an orderly way by putting the number "one" into this new set if the first set does not contain "one" or leaving "one" out if the first set has it, putting the number "two" into the new set if the second set does not have it or leaving "two" out of the new one if that number is in the second set, etc.

The result will be a set of whole numbers that differs from each of the other sets in the collection by at least one number. This new set, of course, must belong to the collection of all sets of whole numbers. But that collection had been put into one-to-one correspondence with the set of all whole numbers (cardinal number, aleph-null) before the new set was constructed.

Therefore, the resultant collection of sets of whole numbers—including the newly constructed set—is larger than the whole number set and must have a higher cardinal number, c, than aleph-null.

Cantor then went on to show that there are infinitely many cardinal numbers that can be arranged in an ascending sequence.

That this is so followed from Cantor's demonstration that the collection of subsets of any set is larger than the set itself. This principle is obvious for finite sets. The subsets of a three-element set of such figures as 1, 2, 3, for example, are 1; 2; 3 (all one-element subsets); 1, 2; 1, 3; 2, 3 (all two-element subsets); 1, 2, 3 (the set itself); and an empty set—or eight in all. Cantor showed this was true for infinite sets as well.



This situation gave rise immediately to a fundamental question involving set theory. If aleph-null is the smallest cardinal number for an infinite set, and there are infinitely many such cardinal numbers, what is the second smallest cardinal number of this sort? Particularly, is c the next higher one after aleph-null?

In other words, is there a collection of things that is larger than the collection of all whole numbers (cardinal number, aleph-null) but smaller than the collection of all sets or the collection of all points on a line (cardinal number, c)? Or is there an intermediate-sized set with a cardinal number between aleph-null and c? This has become known as the "continuum problem" or the "continuum hypothesis."

In 1938, Kurt Goedel of the Institute for Advanced Study in Princeton proved that the continuum hypothesis could not be disproved with the existing axioms of set theory. He did this by showing that the continuum hypothesis was true under a new definition of the word "set" by which all the axioms of set theory held true, also.

This left the question open, however, whether there were other consistent models of set theory in which the continuum hypothesis would be false. In other words, could the continuum problem be solved at all with the axioms of set theory?

It is this fundamental question that Dr. Cohen has now answered. He found another model of sets that satisfied all the axioms of the theory but under which the continuum hypothesis was untrue. Thus, the existence of an intermediate cardinal number between aleph-null and c cannot be proved with set theory as it now stands, and the continuum problem will remain a problem until some fundamentally new axioms are invented.

'Axiom of Choice' Tested

In the course of producing that proof, Dr. Cohen answered another fundamental question. It concerned the "axiom of choice," one of the most important tools in mathematics.

This axiom states that it is possible to take one element out of each set in a collection of sets and make a new set from them.

This may seem on the surface to be self-evident. Yet, it has never been proved, and mathematicians would like to prove it because this axiom is so universally applied. Some years ago, Bertrand Russell gave an illustration of the necessity for the axiom.

One can make a new set of shoes from a set of pairs of shoes by extracting one shoe from each pair without deciding each time which shoe to take: simply order all left shoes (or all right shoes). This cannot be done with a set of pairs of socks, however, because there are no left or right socks. Thus, a choice between the socks in each pair must be made each time for the construction of a new set.

Professor Goedel also worked on this problem and was able to show that the axiom of choice could not be disproved with set theory. Again, however, the question remained whether the axiom could be proved.

This is the second fundamental question that Dr. Cohen has answered. He has shown that the axiom of choice cannot be proved, either, with set theory.

"More important still," Professor Smullyan remarked in this connection, "is that Cohen has shown that the continuum problem cannot be solved even with the use of the axiom of choice."

"Thus," he said, "the most powerful set of reasonable axioms we have at our disposal is insufficient to settle the problem one way or another."

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