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KHAYYAM'S SOLUTION OF CUBIC EQUATIONS

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Omar Khayyam (1044–1123), Persian philosopher and mathematician, had a very interesting geometric solution of the third degree equations. We shall study Khayyam's method and its extension to solving fourth degree equations. This extension is due to M. Hachtroudi, Professor of Mathematics at the University of Teheran. The key to Khayyam's method is the following:

1. THEOREM: *The four points of intersection of two parabolas whose axes are perpendicular are on a circle.*

Proof: For convenience, without loss of generality, we choose the parabolas

$$y^2 = 4p(x - a), \qquad x^2 = 4q(y - b).$$

It is clear that the axes of these parabolas are perpendicular to one another. Now if we add these equations we get

$$x^2 + y^2 - 4px - 4qy + 4ap + 4bq = 0$$

which is a circle with center $(2p, 2q)$. This proves the theorem. Here the complex points of intersection have also been considered. Omar proved this theorem synthetically. We shall leave that to the reader as an exercise.

2. **Solution of cubic equations:** Any third degree equation can be written as

$$x^3 + lx^2 + mx + n = 0. \tag{2.1}$$

If we discuss the solution of a fourth degree equation such as

$$z^4 + az^3 + bz^2 + cz + d = 0, \tag{2.2}$$

then (2.1) will be a special case of (2.2). That is, we consider

$$x^4 + lx^3 + mx^2 + nx = 0. \tag{2.3}$$

Then we ignore the root $x = 0$, and we get the roots of (2.1).

Now let us proceed with the solution of (2.2). If we choose the change of variable $z = x - (a/4)$, the equation (2.2) changes to the form

$$x^4 + Ax^2 + Bx + C = 0. \tag{2.4}$$

We choose $y = x^2$. Then getting the roots of (2.4) is the same as solving the system of equations

$$\begin{cases} x^2 = y \\ y^2 + Ay + Bx + C = 0 \end{cases} \tag{2.5}$$

for x.

It is easily seen that the equations of (2.5) are the equations of two parabolas whose axes are perpendicular to one another. The solution of (2.5) is obtained by the system

$$\begin{cases} x^2 = y \\ x^2 + y^2 + (A - 1)y + Bx + C = 0. \end{cases} \tag{2.6}$$

The advantage of (2.6) is that the parabola $y=x^2$ can be drawn accurately on a sheet of scaled paper. Then a circle of center

$$\left(\frac{A-1}{2}, \frac{B}{2}\right)$$

and radius

$$\frac{\sqrt{A^2+B^2-4AC-2A+1}}{2}$$

can be drawn on a sheet of transparent paper. We superpose the circle on the parabola and read the roots on the x-axis (Fig. 1).

3. An example: The equation

$$x^3+Bx=C \tag{3.1}$$

is solved by Khayyam in the following way. In this equation B and C are supposed positive. We choose b and c such that

$$b^2=B \quad \text{and} \quad b^2c=C.$$

Then (3.1) is written as

$$x^3+b^2x=b^2c.$$

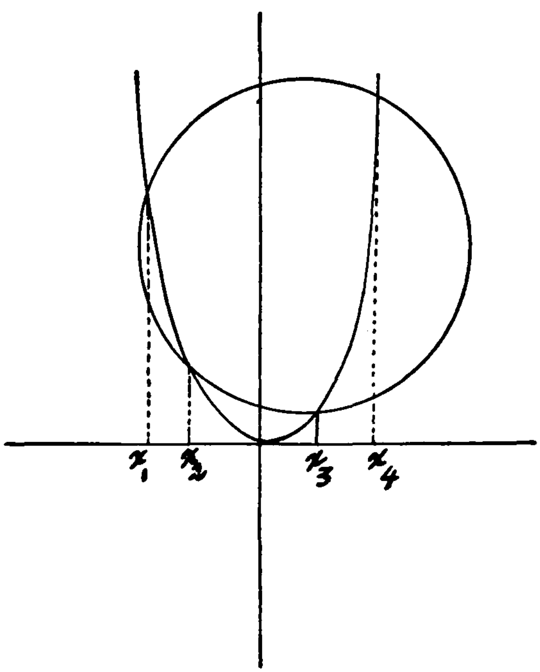

FIG. 1

Choose the segment $AB=b$ and $BC=c$ (Fig. 2). AB is perpendicular to BC.

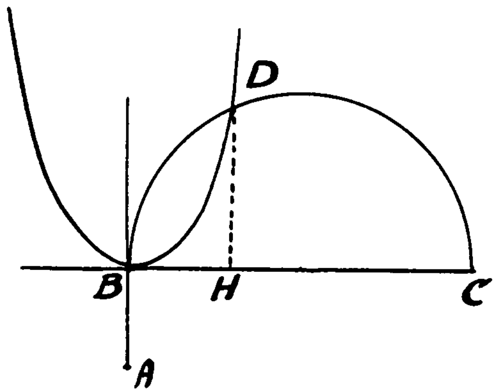

FIG. 2

We draw the half circle with diameter BC, and the parabola with (normal side) AB. This means the parabola $x^2=by$. These curves intersect at D. We draw the perpendicular DH to BC. Then

$$x = DH.$$

Omar proved synthetically that DH is a root of (3.1). The proof is:

$$BH^2 = (AB)(DH).$$

Thus

(3.2) $$\frac{AB}{BH} = \frac{BH}{DH} \quad \text{or} \quad \frac{b}{x} = \frac{x}{DH} \cdot$$

But in the circle

$$\frac{BH}{HD} = \frac{HD}{HC} \cdot$$

Thus

(3.3) $$\frac{AB}{BH} = \frac{DH}{HC}, \quad \text{or} \quad \frac{b}{x} = \frac{DH}{c-x} \cdot$$

Finding DH from (3.2) and (3.3) and comparing them we get

$$x^3 + b^2x = b^2c.$$

Thus the equation has one real root. This root always exists, and is obtained by the intersection of a circle and a parabola.

To explain Omar Khayyam's method we used analytic geometry. Actually in Omar's work every problem was done synthetically.