Omar Khayyam's solution of cubic equations

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4. See R. C. Archibald, "Notes on Omar Khayyam (1050–1122)," *Pi Mu Epsilon Journal* 1 (1953), pp. 351–358.

5. *Discussion of Difficulties of Euclid by Omar Khayyam,* edited with an introduction by Dr. T. Erani, Teheran, 1936.

6. D. E. Smith, "Omar Khayyam and Saccheri," *Scripta Mathematica* 3 (1935), pp. 5–10.

7. On this see also E. B. Plooy, "Euclid's Conception of Ratio and His Definitions of Proportional Magnitudes as Criticized by Arabian Commentators." Thesis, Leiden, 1950, vi+71 pp.

8. On Omar Khayyam as a poet, and an evaluation of the FitzGerald version, see A. J. Arberry, *Omar Khayyam,* Murray, London, 1952, 159 pp.

Omar Khayyam's solution of cubic equations

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Omar Khayyam was the first to solve geometrically, so far as positive roots are concerned, every type of cubic equation. Let us illustrate his procedure for the special type

$$x^3+b^2x+a^3=cx^2,$$

where a, b, c, x are thought of as lengths of line segments. Omar stated this type of cubic rhetorically as "a cube, some sides, and some numbers are equal to some squares." Stated geometrically, the problem of solving the cubic equation is this: Given line segments a, b, c, construct a line segment x such that the above relation among a, b, c, x will hold. The object is to construct x using only straightedge and compasses as far as is possible. A solution using only straightedge and compasses is in general impossible, and at some point of the construction we must be permitted to draw a certain uniquely defined conic section.

A basic construction used several times in the solution of the cubic is that of finding the fourth proportional to three given line segments. This is an old problem whose solution was known to the ancient Greeks. Suppose u, v, w are three given line segments and we desire a line segment x such that $u:v=w:x$. Figure 1 will recall how, with straightedge and compasses, one may construct the desired segment x.

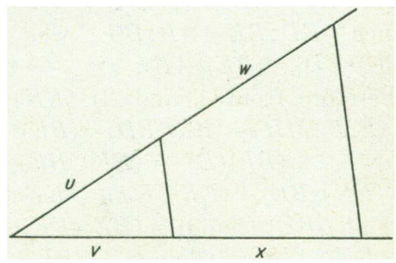

Figure 1

We now follow Omar's geometrical solution of the cubic equation

$$x^3+b^2x+a^3=cx^2.$$

First of all, by the basic construction, find line segment z such that $b:a=a:z$. Then, again by the basic construction, find line segment m such that $b:z=a:m$. We easily find that $m=a^3/b^2$. Now, in Figure 2,

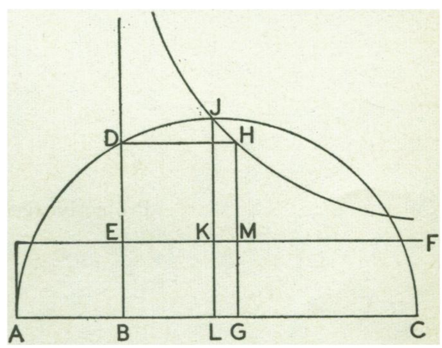

Figure 2

construct $AB=m=a^3/b^2$ and $BC=c$. Draw a semicircle on AC as diameter and let the perpendicular to AC at B cut it in D. On BD mark off $BE=b$ and through E draw EF parallel to AC. By the basic construction, find G on BC such that $ED:BE=AB:BG$ and complete the rectangle $DBGH$. Through H draw the rectangular hyperbola having EF and ED for asymptotes (that is, the hyperbola through H whose equation with respect to EF and ED as x and y axes is of the form $xy=$ a constant). Let the hyperbola cut the semicircle in J, and let the parallel to DE through J cut EF in K and BC in L. Let GH cut EF in M. Now:

1. Since J and H are on the hyperbola, $(EK)(KJ)=(EM)(MH)$.
2. Since $ED:BE=AB:BG$, we have $(BG)(ED)=(BE)(AB)$.
3. Therefore, from (1) and (2), $(EK)(KJ)=(EM)(MH)=(BG)(ED)=(BE)(AB)$.
4. Now $(BL)(LJ)=(EK)(BE+KJ)=(EK)(BE)+(EK)(KJ)=(EK)(BE)+(AB)(BE)$ (by [3]) $=(BE)(EK+AB)=(BE)(AL)$, whence $(BL)^2(LJ)^2=(BE)^2(AL)^2$.
5. But, from elementary geometry, $(LJ)^2=(AL)(LC)$.
6. Therefore, from (4) and (5), $(BE)^2(AL)=(BL)^2(LC)$, or $(BE)^2(BL+AB)=(BL)^2(BC-BL)$.
7. Setting $BE=b$, $AB=a^3/b^2$, $BC=c$ in (6), we obtain $b^2(BL+a^3/b^2)=(BL)^2(c-BL)$.
8. Expanding the last equation in (7), and arranging terms, we find $(BL)^3+b^2(BL)+a^3=c(BL)^2$, and it follows that $BL=x$, a root of the given cubic equation.

It must be admitted that Omar's method is ingenious, and it certainly can interest some high school mathematics students. A point-by-point construction of the hyperbola (utilizing the basic construction) is easy, for if N is any point on EF and if the perpendicular to EF at N cuts the hyperbola in P, then $(EM)(MH)=(EN)(NP)$, whence $EN:EM=MH:NP$, and NP is the fourth proportional to the three given segments EN, EM, MH. In this way a number of points on the hyperbola can be plotted and then the hyperbola sketched in by drawing a smooth curve through the plotted points. The student might be given the numerical cubic $x^3+2x+8=5x^2$. Here $a=2$, $b=\sqrt{2}$, $c=5$. The three roots of this cubic are 2, 4, -1. The student should be able to find the two positive roots by Omar's method; perhaps he can extend the method slightly to find the negative root. Here is the start of an excellent so-called "junior research" project. The student may find Omar's geometrical approach to other types of cubics in J. L. Coolidge, *The Mathematics of Great Amateurs*, Oxford (1950), Chapter II (Omar Khayyam), pp. 19–29. The algebraic solution of cubic equations can be found in any text on the theory of equations.

"Projective geometry is all geometry."—*Cayley.*