Chapter 1

Definitions and Examples

We define Boolean algebras and give some elementary examples.

1. Definition

A Boolean algebra consists of a set B, two binary operations $\wedge$ and $\vee$ (called meet and join respectively), a unary operation $\cdot'$ and two constants $\mathbf{0}$ and $\mathbf{1}$. These obey the following laws:

1. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$;
2. $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$;
3. $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$
4. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$; and
5. $x \wedge x' = \mathbf{0}$ and $x \vee x' = \mathbf{1}$.

2. Examples

Here are some elementary examples; more complicated structures will appear later.

▶ **1**. $\mathcal{P}(\mathbb{N})$ is a Boolean algebra with intersection as meet, union as join, $a' = \mathbb{N} \setminus a$, $\mathbf{0} = \varnothing$ and $\mathbf{1} = \mathbb{N}$.

▶ **2**. Let X be any topological space and let $\mathrm{CO}(X)$ be the family of sets that are simultaneously closed and open (contracted to *clopen*). The family $\mathrm{CO}(X)$ is a Boolean algebra with respect to the same operations as $\mathcal{P}(\mathbb{N})$.

▶ **3**. Let X be a topological space and let $\mathrm{RO}(X)$ be the family of *regular open* sets, i.e., the sets U that satisfy $U = \operatorname{int}\operatorname{cl} U$. For $U, V \in \mathrm{RO}(X)$ define $U \wedge V = U \cap V$, $U \vee V = \operatorname{int}\operatorname{cl}(U \cup V)$, $U' = X \setminus \operatorname{cl} U$, $\mathbf{0} = \varnothing$ and $\mathbf{1} = X$. This makes $\mathrm{RO}(X)$ into a Boolean algebra.

▶ **4**. Let X be any set; the following two families of subsets of X are Boolean algebras.

a. $\mathrm{FC}(X) = \{A \subseteq X : A \text{ is finite or } X \setminus A \text{ is finite}\}$; and

b. $\mathrm{CC}(X) = \{A \subseteq X : A \text{ is countable or } X \setminus A \text{ is countable}\}$.

3. Elementary Properties

One can derive many identities from the laws given above. For example, from Law (3) it follows readily that $x \wedge \mathbf{1} = x \wedge (x \vee x') = x$ and $x \vee \mathbf{0} = x \vee (x \wedge x') = x$; from this we deduce that $x \wedge x = x \wedge (x \vee 0) = x$ and, likewise, $x \vee x = x$.

► **1**. a. If $x \wedge y = \mathbf{0}$ and $x \vee y = \mathbf{1}$ then $y = x'$.
b. De Morgan's laws hold: $(x \wedge y)' = x' \vee y'$ and $(x \vee y)' = x' \wedge y'$.

A Partial Order

A Boolean algebra carries a natural partial order, indiscriminately denoted $\leqslant$, defined by $x \leqslant y$ if $x = x \wedge y$ or, equivalently, $x \vee y = y$.

► **2**. Let B be a Boolean algebra.
a. For all $x, y \in B$ we have $x \wedge y = x$ iff $x \vee y = y$.
b. The relation $\leqslant$ is a partial order on B.
c. For all $x, y \in B$ we have $x \wedge y = \inf\{x, y\}$ and $x \vee y = \sup\{x, y\}$.

Atoms

An *atom* in a Boolean algebra is a non-zero element x with the property that there is no element y with $x > y > \mathbf{0}$.

► **3**. a. The atoms in $\mathrm{CO}(X)$ are the connected clopen sets.
b. If X is Hausdorff then the atoms in $\mathrm{RO}(X)$ correspond to the isolated points of the space X.
c. What are the atoms of $L(H)$?

Chapter 2

Stone's representation theorem

This chapter is devoted to the central result of the course: Stone's representation theorem foor Boolean algebras.

1. Clopen sets

Consider a compact and zero-dimensional Hausdorff space X. We have seen that $\mathrm{CO}(X)$ is a Boolean algebra; what is more remarkable is that X can be recovered from $\mathrm{CO}(X)$. For each $x \in X$ put $u_x = \{C \in \mathrm{CO}(X) : x \in C\}$.

▶ **1**. The family u_x is an *ultrafilter* on $\mathrm{CO}(X)$, i.e.,
a. $\varnothing \notin u_x$;
b. if $C, D \in u_x$ then $C \cap D \in u_x$;
c. if $C \in u_x$ and $C \subseteq D$ then $D \in u_x$; and
d. for every C either $C \in u_x$ or $X \setminus C \in u_x$.

▶ **2**. If u is an ultrafilter on $\mathrm{CO}(X)$ then $u = u_x$ for some x.

▶ **3**. The atoms of $\mathrm{CO}(X)$ correspond to the isolated points of the space X.

Thus X can be seen as the set of ultrafilters on $\mathrm{CO}(X)$. By the definition of zero-dimensionality the family $\mathrm{CO}(X)$ is a base for the topology of X. Therefore X and its topology are completely determined by $\mathrm{CO}(X)$.

2. Stone's Representation Theorem

Stone's Representation Theorem says that every Boolean algebra determines a compact zero-dimensional Hausdorff space.

2.1. Theorem. *To every Boolean algebra B one can associate a compact zero-dimensional space X such that B is isomorphic to the Boolean algebra $\mathrm{CO}(X)$ of clopen subsets of X.*

The space X is denoted $\mathfrak{S}(B)$ and is commonly called the *Stone space* of B. What we have established already is that this association is surjective: every compact zero-dimensional space X is homeomorphic to the Stone space of its own clopen algebra $\mathrm{CO}(X)$.

Ideals and filters

The Stone space is created using ideals and filters on the Boolean algebra. An *ideal* in a Boolean algebra B is a subset I that satisfies

1. $\mathbf{1} \notin I$;
2. if $x, y \in I$ then $x \vee y \in I$; and
3. if $x \in I$ and $y \in B$ then $x \wedge y \in I$.

The first condition is there to prevent B itself from being an ideal; condition (3) can be replaced by: if $x \in I$ and $y \leqslant x$ then $y \in I$.

A *filter* on a Boolean algebra is dual to an ideal, it is a subset F that satisfies

1. $\mathbf{0} \notin F$;
2. if $x, y \in F$ then $x \wedge y \in F$; and
3. if $x \in F$ and $y \in B$ then $x \vee y \in F$.

As above we can replace (3) by: if $x \in F$ and $y \geqslant x$ then $y \in F$.

An *ultrafilter* is a maximal filter, i.e., F is an ultrafilter if it is a filter and every filter G that contains F is equal to F. By Zorn's Lemma every filter is contained in an ultrafilter. The dual to an ultrafilter is a maximal ideal. A consequence of Exercise 1.b is that an ideal I is maximal iff it is *prime*, i.e., iff it satisfies: if $a \wedge b \in I$ then $a \in I$ or $b \in I$.

► **1**. a. If F is a filter then $F' = \{x' : x \in F\}$ is an ideal and vice versa.
b. The following are equivalent for a filter F:
1) F is an ultrafilter;
2) if a is such that $a \wedge b > 0$ for all $b \in F$ then $a \in F$;
3) if $a \vee b \in F$ then $a \in F$ or $b \in F$; and
4) for all $a \in B$ either $a \in F$ or $a' \in F$.
c. A filter F is an ultrafilter iff $F \cup F' = B$.

► **2**. Describe the ultrafilters on $\mathrm{FC}(X)$.

We have seen that the points of a compact zero-dimensional space correspond to the ultrafilters on the Boolean algebra $\mathrm{CO}(X)$. This provides the main inspiration for the definition of $\mathfrak{S}(B)$.

2.2. Definition. The Stone space $\mathfrak{S}(B)$ of the Boolean algebra B has the set of ultrafilters on B as its underlying set. For every $a \in B$ we put

$$\overline{a} = \{u \in \mathfrak{S}(B) : a \in u\}$$

and we use $\{\overline{a} : a \in B\}$ as a base for a topology on $\mathfrak{S}(B)$.

► **3**. Consider the map $a \mapsto \overline{a}$.
a. $\overline{\mathbf{0}} = \varnothing$ and $\overline{\mathbf{1}} = \mathfrak{S}(B)$;
b. $a \neq \mathbf{0}$ iff $\overline{a} \neq \varnothing$;
c. $\overline{a \wedge b} = \overline{a} \cap \overline{b}$;

d. $\overline{a \vee b} = \overline{a} \cup \overline{b}$; and
e. $\overline{a'} = \mathfrak{S}(B) \setminus \overline{a}$.

With this topology $\mathfrak{S}(B)$ becomes a compact Hausdorff space whose clopen set are precisely the sets $\overline{a}$. Once this is established zero-dimensionality of $\mathfrak{S}(B)$ is automatic: the given base consists of clopen sets.

► **4**. Let $\{\overline{a} : a \in A\}$ be a family of clopen sets, such that no finite subfamily covers $\mathfrak{S}(B)$.
a. $I = \bigl\{x : (\exists a_1, \ldots, a_n \in A)(x \leqslant \bigvee_{i=1}^{n} a_i)\bigr\}$ is an ideal in B;
b. if u is any ultrafilter that extends the filter I' then $u \notin \bigcup\{\overline{a} : a \in A\}$;
c. $\mathfrak{S}(B)$ is compact.

► **5**. Let u and v be distinct ultrafilters on B.
a. There is $a \in u \setminus v$;
b. $u \in \overline{a}$ and $v \in \overline{a'}$;
c. $\mathfrak{S}(B)$ is Hausdorff.

► **6**. Each $\overline{a}$ is clopen in $\mathfrak{S}(B)$ and, conversely, if C is a clopen subset of $\mathfrak{S}(B)$ then $C = \overline{a}$ for some a. *Hint*: C is compact, take a finite subcover from the family of basic open sets that are contained in C.

► **7**. Consider the Boolean algebra $\mathrm{FC}(X)$.
a. Is there a topology on X such that $\mathrm{CO}(X) = \mathrm{FC}(X)$?
b. Describe the Stone space of $\mathrm{FC}(X)$.

Chapter 3

Duality

Stone's representation theorem enables one to translate any notion from topology into Boolean algebraic terms and, conversely, any Boolean algebraic notion can be reformulated into something topological. This general conversion process is known as *Stone Duality.*

1. Continuous maps and homomorphisms

The first order of business is to see what happens to continuous maps and homomorphisms.

▶ **1**. Let $f : X \to Y$ be a continuous map between compact Hausdorff spaces.
  a. The map $\mathrm{CO}(f) : \mathrm{CO}(Y) \to \mathrm{CO}(X)$, defined by $\mathrm{CO}(f)(C) = f^{\leftarrow}[C]$, is a homomorphism of Boolean algebras.
  b. If f is onto then $\mathrm{CO}(f)$ is one-to-one.
  c. If f is one-to-one then $\mathrm{CO}(f)$ is onto.

▶ **2**. Let $\varphi : A \to B$ be a homomorphism of Boolean algebras. Define $\mathfrak{S}(\varphi) : \mathfrak{S}(B) \to \mathfrak{S}(A)$ by $\mathfrak{S}(\varphi)(x) = \{a \in A : \varphi(a) \in x\}$.
  a. $\mathfrak{S}(\varphi)$ is a well-defined.
  b. $\mathfrak{S}(\varphi)$ is continuous.
  c. If φ is onto then $\mathfrak{S}(\varphi)$ is one-to-one.
  d. If φ is one-to-one then $\mathfrak{S}(\varphi)$ is onto.

▶ **3**. a. If O is an open subset of X then $\{C \in \mathrm{CO}(X) : C \subseteq O\}$ is an ideal in $\mathrm{CO}(X)$.
  b. If I is an ideal in B then $\bigcup\{\overline{a} : a \in I\}$ is an open subset of $\mathfrak{S}(B)$.

▶ **4**. a. If F is an closed subset of X then $\{C \in \mathrm{CO}(X) : F \subseteq C\}$ is a filter on $\mathrm{CO}(X)$.
  b. If F is a filter on B then $\bigcap\{\overline{a} : a \in F\}$ is an closed subset of $\mathfrak{S}(B)$.

▶ **5**. Let Y be a closed subspace of the compact zero-dimensional Hausdorff space X. If $f : Y \to X$ is the embedding then the dual map $\mathrm{CO}(f)$ is defined by $C \mapsto C \cap Y$. The kernel of $\mathrm{CO}(f)$ is the ideal $I = \{C : C \cap F = \varnothing\}$. The algebra $\mathrm{CO}(Y)$ is isomorphic to the quotient $\mathrm{CO}(X)/I$.

Chapter 4

Specific examples

Although in general it is hard to get a concrete description of the Stone space of a Boolean algebra there are some cases where we can really see what it looks like.

1. Interval algebras

Linear orders offer a rich supply of Boolean algebras.

► **1**. Let B_I be the Boolean subalgebra of $\mathcal{P}(\mathbb{R})$ generated by the family $\bigl\{(-\infty, r] : r \in \mathbb{R}\bigr\}$, the so-called *interval algebra* of $\mathbb{R}$.

a. The families $\bigl\{(-\infty, r] : r \in \mathbb{R}\bigr\}$ and $\bigl\{(r, \infty) : r \in \mathbb{R}\bigr\}$ determine ultrafilters on B_I. Denote them by $u_{-\infty}$ and u_∞ respectively.

b. For every r the families $\bigl\{(q, r] : q < r\bigr\}$ and $\bigl\{(r, s] : s > r\bigr\}$ determine ultrafilters on B_I. Denote them by u_r^- and u_r^+ respectively.

c. Every ultrafilter on B_I is of one of the aforementioned forms. *Hint*: If $u \neq u_{-\infty}, u_\infty$ consider $r = \inf\{s : (-\infty, s] \in u\}$. Then also $r = \sup\{q : (q, \infty) \in u\}$. If $(-\infty, r] \in u$ then $u = u_r^-$; If $(r, \infty) \in u$ then $u = u_r^+$.

d. The topology of $\mathfrak{S}(B_I)$ is the topology induced by the linear order $\prec$ that is defined by $u_{-\infty} \prec u_q^+ \prec u_r^- \prec u_r^+ \prec u_s^- \prec u_\infty$ whenever $q < r < s$.
Hint: $\overline{(-\infty, r]} = [u_{-\infty}, u_r^-]$ and $\overline{(r, \infty)}, = [u_r^+, u_\infty]$.

e. The subspaces $\{u_r^- : r \in \mathbb{R}\}$ and $\{u_r^+ : r \in \mathbb{R}\}$ are homeomorphic to the well-known Sorgenfrey line.

The space $\mathfrak{S}(B_I)$ is known as *Alexandroff's double arrow space*. We shall use $\mathbb{S} = \{u_r^+ : r \in \mathbb{R}\}$ as our incarnation of the Sorgenfrey line, or rather the real line with the topology generated by the intervals of the form $(a, b]$.

One can use any linear order in this construction; one gets the class of Boolean algebras known as *interval algebras*.

2. $\mathcal{P}(\mathbb{N})$ and some of its subfamilies

The set $\mathbb{N}$ of natural numbers and its power set $\mathcal{P}(\mathbb{N})$ are basic in Set-Theoretic Topology. As explained in Appendix B we take the set-theoretical definition of $\mathbb{N}$ as the minimal inductive set. This implies that our notation becomes very economical: every element n is a subset of $\mathbb{N}$ and arithmetic has been set up in such a way that $n = \{i \in \mathbb{N} : i < n\}$ for every n. Thus we can take initial segments of sequences simply by taking restrictions: if $x = \langle x_n \rangle_n$ is a sequence then $x \restriction 3 = \langle x_0, x_1, x_2 \rangle$.

As we shall see many constructions employ sets or families of sets of natural numbers. Using various incarnations of $\mathbb{N}$ one can construct many useful, and sometimes unexpected, families of subsets of $\mathbb{N}$. For example, it would seem that a chain of subsets of $\mathbb{N}$ must be countable because $\mathbb{N}$ itself is countable: distributing $\mathbb{N}$ over uncountably many difference sets seems impossible. The possible misconception here is that the difference sets must be neatly arranged in some sort of disjoint family and because disjoint families of subsets of $\mathbb{N}$ are countable so is our chain. This misconception can be dispelled by identifying $\mathbb{N}$ with the countable set $\mathbb{Q}$ of rational numbers, for we can then exhibit a chain of subsets of $\mathbb{N}$ that is order-isomorphic to $\mathbb{R}$: put $Q_x = \{q \in \mathbb{Q} : q \leqslant x\}$ for every x.

A very useful concept is that of *almost disjointness*; we say that two subsets of $\mathbb{N}$ are almost disjoint if their intersection is finite. The difference between disjointness and almost disjointness is perhaps best illustrated by considering the set ${}^{<\mathbb{N}}2$ of finite sequences of zeros and ones. Thus, $s \in {}^{<\mathbb{N}}2$ means there is an $n \in \mathbb{N}$ such that $s : n \to 2$. If we let ${}^{n}2$ denote the set of functions from n to 2 then evidently ${}^{<\mathbb{N}}2 = \bigcup_{n \in \mathbb{N}} {}^{n}2$. This makes it clear that ${}^{<\mathbb{N}}2$ is countable — each ${}^{n}2$ is finite. The set ${}^{<\mathbb{N}}2$ is naturally ordered by extension of functions: $s \leqslant t$ means $\operatorname{dom} s \subseteq \operatorname{dom} t$ and $s(i) = t(i)$ for all $i \in \operatorname{dom} s$. As such it looks like a *tree*: if $s \in {}^{<\mathbb{N}}2$ with $\operatorname{dom} s = n$ and $t \leqslant s$ then $t = s \restriction m$ for some $m \leqslant n$. A *branch* through this tree is a sequence $\langle s_n \rangle_n$ in ${}^{<\mathbb{N}}2$ such that $\operatorname{dom} s_n = n$ and $s_n \leqslant s_{n+1}$ for all n. Such a branch determines an element of ${}^{\mathbb{N}}2$: put $x = \bigcup_n s_n$. Conversely any element x of ${}^{\mathbb{N}}2$ determines the branch $\langle x \restriction n \rangle_n$.

For $x \in {}^{\omega}2$ put $B_x = \{x \restriction n : n \in \omega\}$. If $x \neq y$ then $B_x \cap B_y$ is finite: let n be minimal with $x(n) \neq y(n)$, then $|B_x \cap B_y| = n$. We see that $\{B_x : x \in {}^{\mathbb{N}}2\}$ is an almost disjoint family that is just as big as $\mathcal{P}(\mathbb{N})$ is.

The exercises contain some more large and complicated families of subsets of $\mathbb{N}$; each of them shall, at one time or another, be put to good use.

► **1** (SIERPIŃSKI [**11**]). For every irrational number x put $S_x = \bigl\{\frac{1}{n}\lfloor nx \rfloor : n \in \mathbb{N}\bigr\}$. Then $\{S_x : x \in \mathbb{P}\}$ is an almost disjoint family (in fact if $x < y$ then $|S_x \cap S_y| < \frac{1}{y-x}$).

► **2** (HAUSDORFF [**3**]). Put $C = \bigl\{\langle s, n \rangle : n \in \mathbb{N}, s \subseteq \mathcal{P}(n)\bigr\}$ and for $x \in \mathcal{P}(\mathbb{N})$ put

$$I_x = \bigl\{\langle s, n \rangle : x \cap n \in s\bigr\}.$$

The set C is countable and the family $\{I_x : x \in \mathcal{P}(\mathbb{N})\}$ is *independent*, i.e., if $x_1, \ldots, x_m, y_1, \ldots, y_n$ are all distinct then $\bigcap_{i=1}^{m} I_{x_i} \setminus \bigcup_{j=1}^{n} I_{y_j}$ is infinite.

► **3** (KUNEN [**5**]). Put $D = \bigl\{\langle f, n \rangle : n \in \mathbb{N}, f \in {}^{\mathcal{P}(n)}\mathcal{P}(n)\bigr\}$ and for $x, y \in \mathcal{P}(\mathbb{N})$ put

$$I_{x,y} = \bigl\{\langle p, n \rangle : p(x \cap n) = y \cap n\bigr\}.$$

The set D is countable and $\{I_{x,y} : x, y \in \mathcal{P}(\mathbb{N})\}$ is an *independent matrix*, i.e.,

a. for every x the family $\{I_{x,y} : y \in \mathcal{P}(\mathbb{N})\}$ is almost disjoint; and

b. if F is a finite subset of $\mathcal{P}(\mathbb{N})$ and $f : F \to \mathcal{P}(\mathbb{N})$ is a function then the intersection $\bigcap_{x\in F} I_{x,f(x)}$ is infinite.

▶ **4** (Kunen [**6**], van Mill [**13**]). Put $E = \{\langle p, n\rangle : n \in \mathbb{N}, p \in {}^{\mathcal{P}(n)}\mathcal{P}(\mathcal{P}(n))\}$ and for $x, y \in \mathcal{P}(\mathbb{N})$ and $m \in \mathbb{N}$ put

$$I_{x,y,m} = \{\langle p, n\rangle : y \cap n \in p(x \cap n) \text{ and } |p(x \cap n)| \leqslant m\}.$$

The set E is countable and $\{I_{x,y,m} : x, y \in \mathcal{P}(\mathbb{N}), m \in \mathbb{N}\}$ is an *independent linked family*, i.e.,

a. for fixed x and y and for every m we have $I_{x,y,m} \subseteq I_{x,y,m+1}$;

b. for fixed x and m the family $\{I_{x,y,m} : y \in \mathcal{P}(\mathbb{N})\}$ is *precisely m-linked*, i.e., whenever $F \subseteq \mathcal{P}(\mathbb{N})$ has m (or fewer) elements then $\bigcap_{y\in F} I_{x,y,m}$ is infinite but if it has $m + 1$ (or more) elements then $\bigcap_{y\in F} I_{x,y,m}$ is finite; and

c. whenever we take a finite subset F of $\mathcal{P}(\mathbb{N})$ and for each $x \in F$ a natural number m_x and a subset σ_x of $\mathcal{P}(\mathbb{N})$ with $|\sigma_x| = m_x$ the intersection

$$\bigcap_{x\in F} \bigcap_{y\in\sigma_x} I_{x,y,m_x} \qquad (*)$$

is infinite.

d. Define $\pi : E \to \mathbb{N}$ by $\pi(p, n) = n$; the image of $(*)$ under π is a cofinite subset of $\mathbb{N}$.

3. More Boolean algebras

Using the families of subsets of $\mathbb{N}$ that we have seen above we can define more Boolean algebras and calculate their Stone spaces.

▶ **1**. Let $\mathbb{B}_{\mathrm{ad}}$ be the Boolean subalgebra of $\mathcal{P}({}^{<\mathbb{N}}2)$ generated by the almost disjoint family defined in Section 2.

a. Every finite set belongs to $\mathbb{B}_{\mathrm{ad}}$.

b. For every $s \in {}^{<\mathbb{N}}2$ the set $u_s = \{b : s \in b\}$ is an ultrafilter on $\mathbb{B}_{\mathrm{ad}}$.

c. For every $x \in {}^{\mathbb{N}}2$ the family $u_x = \{b : B_x \setminus b \text{ is finite}\}$ is an ultrafilter on $\mathbb{B}_{\mathrm{ad}}$.

d. The family $\{{}^{<\mathbb{N}}2 \setminus B_x : x \in {}^{\mathbb{N}}2\}$ determines an ultrafilter u_∞ on $\mathbb{B}_{\mathrm{ad}}$.

e. Every ultrafilter on $\mathbb{B}_{\mathrm{ad}}$ is of the form u_s, u_x or u_∞. *Hint*: If $\{s\} \in u$ for some s then $u = u_s$, otherwise if $B_x \in u$ for some x then $u = u_x$, otherwise $u = u_\infty$.

f. Let $X = {}^{<\mathbb{N}}2 \cup {}^{\mathbb{N}}2 \cup \{\infty\}$, topologized by making every point of ${}^{<\mathbb{N}}2$ isolated; using the sets $\{x\} \cup \{x \restriction n : n \geqslant m\}$ as basic neighbourhoods at $x \in {}^{\omega}2$; and by using the sets $X \setminus (B_x \cup \{x\})$ as a local subbase at ∞. Then X is homeomorphic to $\mathfrak{S}(\mathbb{B}_{\mathrm{ad}})$.

g. The subspace $X \setminus \{\infty\}$ of X is not normal. *Hint*: Consider the closed sets $P = \{x \in {}^{\mathbb{N}}2 : x^{\leftarrow}(0) \text{ and } x^{\leftarrow}(1) \text{ are infinite}\}$ and $Q = {}^{\mathbb{N}}2 \setminus P$.

The subspace ${}^{<\mathbb{N}}2 \cup {}^{\mathbb{N}}2$ of $\mathfrak{S}(\mathbb{B}_{\mathrm{ad}})$ is known as the *Cantor tree*.

definition of subbase?

► **2**. Let $\mathcal{I}$ be an independent family in $\mathcal{P}(\mathbb{N})$ indexed by $\mathcal{P}(\mathbb{N})$ itself, as in Exercise 2 of Section 2, and let $\mathbb{B}_{\mathrm{if}}$ be the Boolean algebra generated by $\mathcal{I}$. For $x \in \mathcal{P}(\mathbb{N})$ put $I_{x,1} = I_x$ and $I_{x,0} = \mathbb{N} \setminus I_x$.

a. Every function $t : \mathcal{P}(\mathbb{N}) \to 2$ determines an ultrafilter u_t on $\mathbb{B}_{\mathrm{if}}$; it has $\{I_{x,t(x)} : x \in \mathcal{P}(\mathbb{N})\}$ as a subbase.

b. If u is an ultrafilter on $\mathbb{B}_{\mathrm{if}}$ then $u = u_t$, where $t(x) = 0$ if $I_x \notin u$ and $t(x) = 1$ if $I_x \in u$.

c. $\mathfrak{S}(\mathbb{B}_{\mathrm{if}})$ is homeomorphic to the product ${}^{\mathcal{P}(\mathbb{N})}2$. *Hint*: $\overline{I_{x,1}} = \{u_t : t(x) = 1\}$ and $\overline{I_{x,0}} = \{u_t : t(x) = 0\}$.

d. For $n \in \mathbb{N}$ define t_n by $t_n(x) = 0$ if $n \notin I_x$ and $t_n(x) = 1$ if $n \in I_x$; the set $\{t_n : n \in \mathbb{N}\}$ is dense in ${}^{\mathcal{P}(\mathbb{N})}2$.

► **3**. Let $\mathbb{B}'_{\mathrm{if}}$ be the Boolean algebra generated by $\mathbb{B}_{\mathrm{if}}$ and the finite subsets of $\mathbb{N}$; describe $\mathfrak{S}(\mathbb{B}_{\mathrm{if}})$.

Chapter 5

$\mathbb{M}$ and $\mathcal{P}(\mathbb{N})$

We discuss two important algebras: the measure algebra and the power set of $\mathbb{N}$.

1. The Measure algebra

The family $\mathcal{B}$ of Borel subsets of $[0,1]$ is a Boolean algebra and the family $\mathcal{N}$ of sets of measure zero is an ideal in $\mathcal{B}$. The quotient algebra $\mathbb{M} = \mathcal{B}/\mathcal{N}$ is known as the *measure algebra*. We let $q : \mathcal{B} \to \mathbb{M}$ denote the quotient homomorphism.

▶ **1**. a. If A is a countable subset of $\mathcal{B}$ then $\bigvee q[A] = q[\bigcup A]$.
b. Every set of pairwse disjoint elements of $\mathbb{M}$ is countable.
c. The Boolean algebra $\mathbb{M}$ is complete. *Hint*: If $A \subseteq \mathbb{M}$ let B be a maximal pairwise disjoint subset of $\{b : (\exists a \in A)(b \leqslant a)\}$; then $\bigvee B = \bigvee A$.
d. Every ultrafilter on $\mathbb{M}$ has elements of arbitrarily small measure.
e. For every countable set A of ultrafilters on $\mathbb{M}$ there is a nonzero element a of $\mathbb{M}$ such that $a \notin u$ for all $u \in A$.
f. The Stone space $\mathfrak{S}(\mathbb{M})$ of $\mathbb{M}$ is extremally disconnected, is not separable and has no uncountable disjoint family of open sets.

2. The algebra $\mathcal{P}(\mathbb{N})$

The Stone space of $\mathcal{P}(\mathbb{N})$ is usually denoted $\beta\mathbb{N}$.

The atoms of $\mathcal{P}(\mathbb{N})$ are the singleton sets $\{n\}$. These give us countably many isolated points in $\beta\mathbb{N}$; we generally identify n with the isolated point of $\beta\mathbb{N}$ determined by the atom $\{n\}$. Thus we treat $\mathbb{N}$ as a subset of $\beta\mathbb{N}$.

The other points are not easy to describe; ultrafilters on $\mathbb{N}$ are in a certain sense indescribable: if $u \in \beta\mathbb{N} \setminus \mathbb{N}$ then $\{r_x : x \in u\}$ is a nonmeasurable subset of $\mathbb{R}$, where $r_x = \sum_{n \in x} 2^{-n}$. This means that we cannot hope to describe $\beta\mathbb{N}$ like we did $\mathfrak{S}(\mathbb{B}_{\mathrm{ad}})$ or $\mathfrak{S}(\mathbb{B}_{\mathrm{if}})$. Nevertheless, as we shall see there is a rich variety of points in $\beta\mathbb{N} \setminus \mathbb{N}$; we shall merely have to work a lot harder to describe them. For now we content ourselves with deducing some global properties of $\beta\mathbb{N}$ and $\beta\mathbb{N} \setminus \mathbb{N}$ that require little beyond the knowledge that we acquired thus far.

▶ **1**. The set $\mathbb{N}$ is dense in $\beta\mathbb{N}$. *Hint*: $\overline{a} \cap \mathbb{N} = a$.

► **2**. If $a \subseteq \mathbb{N}$ then $\operatorname{cl} a = \overline{a}$. *Hint*: $\operatorname{cl} a \subseteq \overline{a}$, $\overline{a} \cap \overline{a'} = \varnothing$ and $\operatorname{cl} a \cup \operatorname{cl} a' = \operatorname{cl} \mathbb{N} = \beta\mathbb{N}$.

► **3**. The space $\beta\mathbb{N}$ is extremally disconnected, i.e., if U is open then $\operatorname{cl} U$ is open or, equivalently, if U and V are open and disjoint then $\operatorname{cl} U \cap \operatorname{cl} V = \varnothing$.
Hint: $\operatorname{cl} U = \operatorname{cl}(U \cap \mathbb{N}) = \overline{U \cap \mathbb{N}}$.

We denote the subspace $\beta\mathbb{N} \setminus \mathbb{N}$ by $\mathbb{N}^*$ and generally we put $a^* = \overline{a} \setminus \mathbb{N}$. By Exercise 5 every clopen subset of $\mathbb{N}^*$ is of the form a^* and $\mathbb{N}^*$ is the Stone space of the algebra $\mathcal{P}(\mathbb{N})/\mathit{fin}$, where *fin* denotes the ideal of finite subsets of $\mathbb{N}$.

► **4**. Let $a, b \in \mathcal{P}(\mathbb{N})$.
a. a^* and b^* are disjoint iff a and b are almost disjoint.
b. $a^* \subseteq b^*$ iff $a \setminus b$ is finite.

► **5**. The dual of the embedding of $\mathbb{B}_{\mathrm{ad}}$ into $\mathcal{P}(\mathbb{N})$ is a continuous map f of $\beta\mathbb{N}$ onto $\mathfrak{S}(\mathbb{B}_{\mathrm{ad}})$.
a. If $s \in {}^{<\mathbb{N}}2$ then $f(s) = s$.
b. If $u \in B_x^*$ then $f(u) = x$.
c. If $u \in \mathbb{N}^* \setminus \bigcup_x B_x^*$ then $f(u) = \infty$.
d. The family $\{B_x^* : x \in {}^{\mathbb{N}}2\}$ is a pairwise disjoint family of open sets in $\mathbb{N}^*$.
e. The space $\mathbb{N}^*$ is not extremally disconnected. *Hint*: Refer to Exercise 1.g and let $U = \bigcup_{x \in P} B_x^*$ and $V = \bigcup_{x \in Q} B_x^*$; then $\operatorname{cl} U \cap \operatorname{cl} V \neq \varnothing$.

► **6**. a. There is a continuous map from $\beta\mathbb{N}$ onto ${}^{\mathcal{P}(\mathbb{N})}2$. *Hint*: Consider the embedding of $\mathbb{B}_{\mathrm{if}}$ into $\mathcal{P}(\mathbb{N})$.
b. Investigate the natural continuous map from $\beta\mathbb{N}$ onto $\mathfrak{S}(\mathbb{B}'_{\mathrm{if}})$.

► **7**. For $n \in \mathbb{N}$ define $s_n \in {}^{\mathcal{P}(\mathbb{N})}2$ by $s_n(x) = 0$ if $n \notin x$ and $s_n(x) = 1$ if $n \in x$. Then $\operatorname{cl}\{s_n : n \in \mathbb{N}\}$ is homeomorphic to $\beta\mathbb{N}$. Deduce that $\beta\mathbb{N}$ and ${}^{\mathcal{P}(\mathbb{N})}2$ have the same number of points.

To indicate that a relation holds except for finitely many exceptions we adorn it with a star. Thus, $a \subseteq^* b$ means that $a \subseteq b$ with possibly finitely many points of a not belonging to b, in other words that $a \setminus b$ is finite; we have seen that this is equivalent to $a^* \subseteq b^*$ (hence the star). Extending this, $a \subset^* b$ means $a \subseteq^* b$ but $b \setminus a$ is infinite, and $a \cap b =^* \varnothing$ means that a and b are almost disjoint.

► **8**. Let $\langle a_n \rangle_n$ be a sequence in $\mathcal{P}(\mathbb{N})$ such that $a_n \subset^* a_{n+1}$ for all n. There is a set $a \in \mathcal{P}(\mathbb{N})$ such that $a_n \subset^* a$ for all n and $a \subset^* \mathbb{N}$. *Hint*: Note that $a_n \setminus \bigcup_{m<n} a_m$ is always infinite; pick k_n in this difference and let $a = \mathbb{N} \setminus \{k_n : n \in \mathbb{N}\}$.

► **9**. Let $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ be two sequences in $\mathcal{P}(\mathbb{N})$ such that $a_n \subset^* a_{n+1} \subset^* b_{n+1} \subset^* b_n$ for all n. There is a set $c \in \mathcal{P}(\mathbb{N})$ such that $a_n \subset^* c \subset^* b_n$ for all n.
Hint: Consider $c = \bigcup_{n \in \mathbb{N}} (a_n \cap \bigcap_{m \leqslant n} b_m)$.

► **10**. a. If G is a nonempty G_δ-subset of $\mathbb{N}^*$ then $\operatorname{int} G \neq \varnothing$.
b. If U is an open F_σ-subset of $\mathbb{N}^*$ then $\operatorname{cl} U$ is not open (unless $U = \mathbb{N}^*$).
c. If U and V are disjoint open F_σ-subsets of $\mathbb{N}^*$ then $\operatorname{cl} U \cap \operatorname{cl} V = \varnothing$.

Chapter 6

Completeness and completions

We have already encountered complete Boolean algebras. Here we properly define completeness and study the notion in more detail.

1. Completeness

A Boolean algebra is *complete* if every subset of it has a supremum. If B is a Boolean algebra and $A \subseteq B$ then we write $\bigvee A$ for the supremum of A and $\bigwedge A$ for its infimum.

▶ **1**. The following Boolean algebras are complete.
a. $\mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$;
b. $\mathbb{M}$, the measure algebra;
c. $\mathrm{RO}(X)$, the regular-open algebra of a topological space X.

▶ **2**. The quotient-algebra $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is not complete.

The dual of completeness is extremal disconnectedness. A topological space is *extremally disconnected* if the closure of every open set is open.

▶ **3**. A space is extremally disconnected iff disjoint open sets have disjoint closures.

▶ **4**. If B is a complete Boolean algebra then $\mathfrak{S}(B)$ is extremally disconnected. *Hint*: If U is open then $\operatorname{cl} U = \overline{a}$, where $a = \bigwedge\{b : \bar{b} \subseteq U\}$.

▶ **5**. If X is an extremally disconnected Hausdorff space then $\mathrm{CO}(X)$ is complete. *Hint*: Show that $\mathrm{CO}(X) = \mathrm{RO}(X)$.

2. Completion

Let B be any Boolean algebra and put $\bar{B} = \mathrm{RO}(\mathfrak{S}(B))$. We have seen that $\bar{B}$ is complete; it is also clear that B is a subalgebra of $\bar{B}$.

▶ **1**. The algebra B is *dense* in $\bar{B}$, i.e., if $a \in \bar{B}$ and $a > \mathbf{0}$ then there is $b \in B$ such that $a \geqslant b > \mathbf{0}$.

We call $\bar{B}$ the *completion* of B, because of the following result.

▶ **2**. If C is a complete Boolean algebra that contains B as a dense subalgebra then there is an isomorphism $\varphi : C \to \bar{B}$ such that $\varphi(a) = a$ for $a \in B$.

3. The absolute

The dual notion of completion is that of the *absolute* of a space. The absolute can be defined for arbitrary compact Hausdorff spaces.

Let X be compact Hausdorff and denote $E(X) = \mathfrak{S}(\mathrm{RO}(X))$. Define $\pi_X : E(X) \to X$ by

$$\bigl\{\pi_X(u)\bigr\} = \bigcap\{\mathrm{cl}\, U : U \in u\}.$$

The space $E(X)$ — or better the pair $\bigl(E(X), \pi_X\bigr)$ — is called the *absolute* or *Gleason space* of X.

► **1**. a. The map π_X is well-defined.
b. The map π_X is continuous and onto.
c. The map π_X is irreducible, i.e., if $F \subseteq X$ is closed and $\pi_X[F] = X$ then $F = E(X)$.

The absolute has a certain universality property: it is the largest irreducible preimage of the space.

► **2**. Let X and Y be compact Hausdorff spaces and let $f : Y \to X$ be continuous, irreducible and onto. There is a map $g : E(X) \to Y$ such that $f \circ g = \pi_X$. *Hint*: The map $U \mapsto f^{\leftarrow}[U]$ is an isomorphism between $\mathrm{RO}(X)$ and $\mathrm{RO}(Y)$; define g by $\bigl\{g(u)\bigr\} = \bigcap\bigl\{\mathrm{cl}\, f^{\leftarrow}[U] : U \in u\bigr\}$.

The same result can be proved for maps that are not necessarily irreducible.

► **3**. Let X and Y be compact Hausdorff spaces and let $f : Y \to X$ be continuous and onto. There is a map $g : E(X) \to Y$ such that $f \circ g = \pi_X$. *Hint*: Apply Zorn's Lemma to find a closed subset Z of Y such that $f \restriction Z$ is irreducible.

► **4**. Let $f : X \to Y$ be a continuous, irreducible and onto map between compact Hausdorff spaces, where Y is extremally disconnected. Then f is a homeomorphism. *Hint*: $Y = E(Y)$, apply Exercise 2.

► **5**. If $f : X \to X$ is a homeomorphism of the compact Hausdorff space X then $E(f)(u) = \{f[U] : U \in u\}$ defines a homeomorphism of $E(X)$ such that $f \circ \pi_X = \pi_X \circ E(f)$.

We now describe a strange subspace of $E({}^{\omega}2)$. Fix $u_0 \in E({}^{\omega}2)$ such that $\pi_{{}^{\omega}2}(u_0)$ is the point with all coordinates zero. Note that we can use addition modulo 2 to make for every $x \in {}^{\omega}2$ the point $u_x = \{U + x : U \in u_0\}$. We let $A = \{u_x : x \in {}^{\omega}2\}$.

► **6**. a. The space A is dense in $E({}^{\omega}2)$.
b. The space A is separable.
c. The space A is homogeneous.

d. If u_0 has the property that $u_0 \notin \operatorname{cl} D$, whenever D is nowhere dense and $u_0 \notin D$, then every nowhere dense set is closed in A. (Later we shall construct such a point in $E({}^{\omega}2)$.

4. Extremally disconnected spaces

We derive some more properties of extremally disconnected spaces.

▶ **1**. a. Every dense subspace of an extremally disconnected space is extremally disconnected.

b. Every open subspace an extremally disconnected space is extremally disconnected.

c. Closed subspaces of extremally disconnected spaces need not be extremally disconnected.

d. Closures of open subspaces of extremally disconnected spaces are extremally disconnected.

▶ **2**. Let X be an extremally disconnected compact Hausdorff space. If D is open or dense in X then every continuous function $f : D \to [0,1]$ can be extended to a continuous function $\tilde{f} : X \to [0,1]$. *Hint*: If D is dense define $\tilde{f}(x) = \inf\{r : x \in \operatorname{cl} f^{\leftarrow}[0,r]\}$. If D is open then D is dense in $\operatorname{cl} D$, which is extremally disconnected; then apply the Tietze-Urysohn theorem.

Chapter 7

Uniqueness results

In this chapter we show how two spaces are characterized by a short list of topological properties. These spaces are the Cantor set and the space $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$.

1. The Cantor set

It is well-known that Cantor's middle-third set is homeomorphic to the product space ${}^{\omega}2$.

▶ **1**. Prove that each of the following formulas defines a metric on ${}^{\omega}2$ that generates its product topology.

a. $d_1(x, y) = \sum_{i=0}^{\infty} 2^{-i}|x_i - y_i|$;

b. $d_2(x, y) = 2^{-\Delta(x,y)}$ if $x \neq y$ and $d_2(x, x) = 0$. Here $\Delta(x, y) = \min\{i : x_i \neq y_i\}$. This metric satisfies the *strong triangle inequality*: $d_2(x, z) \leqslant \max\bigl\{d_2(x, y), d_2(y, z)\bigr\}$.

▶ **2**. Define $f : {}^{\omega}2 \to [0, 1]$ by $f(x) = 2\sum_{i=0}^{\infty} \frac{x_i}{3^{i+1}}$.

a. Verify that $f(x) \in [0, 1]$ for all x.

b. Prove that $f[{}^{\omega}2]$ is Cantor's middle-third set.

c. The map f is one-to-one and continuous.

d. The map f is a homeomorphism.

In 1910 L. E. J. Brouwer gave the following characterization of the Cantor set.

1.1. Theorem. *If X is a compact and metrizable zero-dimensional space without isolated points then X is homeomorphic to the Cantor set.*

We shall prove this theorem via the Boolean algebra of clopen sets of X. We still have to translate metrizability into Boolean algebraic terms.

▶ **3**. A compact zero-dimensional space is metrizable iff its Boolean algebra of clopen sets is countable.

a. If $\mathrm{CO}(X)$ is countable then X is metrizable. *Hint*: Enumerate $\mathrm{CO}(X)$ as $\{C_n : n \in \omega\}$ and define $f : X \to {}^{\omega}2$ by $f(x)_i = 1$ iff $x \in C_i$. Then f is an embedding.

b. If X is metrizable then $\mathrm{CO}(X)$ is countable. *Hint*: X has a countable base and every $C \in \mathrm{CO}(X)$ is a finite union of basic open sets.

Now we can formulate the dual of Theorem 1.1.

1.2. Theorem. *Any two countable Boolean algebras without atoms are isomorphic.*

► **4**. Let A be a finite Boolean algebra.

a. Below every non-zero element of A there is an atom. *Hint*: If $a > \mathbf{0}$ take an ultrafilter u with $a \in u$; consider $\bigwedge u$.

b. If $a > \mathbf{0}$ then $a = \bigvee\{b : b \leqslant a \text{ and } b \text{ is an atom}\}$.

We denote the set of atoms of a Boolean algebra B by $\mathrm{At}(B)$.

► **5**. Prove Theorem 1.2. Let A and B be two countable Boolean algebras without atoms and enumerate them as $\{a_n : n \in \omega\}$ and $\{b_n : n \in \omega\}$ respectively. Recursively define isomorphisms $\varphi_n : A_n \to B_n$, where A_n and B_n are finite subalgebras of A and B respectively and such that $\{a_i : i < n\} \subseteq A_n$ and $\{b_i : i < n\} \subseteq B_n$ as well as $\varphi_n \subseteq \varphi_{n+1}$ for all n.

a. Put $A_0 = \{\mathbf{0}, \mathbf{1}\} = B_0$ and define $\varphi(\mathbf{0}) = \mathbf{0}$ and $\varphi(\mathbf{1}) = \mathbf{1}$.

b. At stage n divide $\mathrm{At}(A_n)$ into three sets $L = \{a : a \leqslant a_n\}$, $D = \{a : a \wedge a_n = \mathbf{0}\}$ and $S = \mathrm{At}(A) \setminus (L \cup D)$. The value $b = \varphi_{n+1}(a_n)$ should satisfy: $(a \in L) \to \bigl(\varphi_n(a) \leqslant b\bigr)$, $(a \in D) \to \bigl(\varphi_n(a) \wedge b = \mathbf{0}\bigr)$ and $(a \in S) \to \bigl((\varphi_n(a_n) \wedge b > \mathbf{0}) \wedge (\varphi_n(a_n) \wedge b' > \mathbf{0})\bigr)$.

c. For every $a \in S$ there is $b_a \in B$ such that $\mathbf{0} < b_a < \varphi_n(a)$.

d. Let A_n^+ be the Boolean algebra generated by A_n and a_n. Define $\varphi_n^+ : A_n' \to B$ by putting $\varphi_n^+(a) = \varphi_n(a)$ for $a \in L \cup D$ as well as $\varphi_n^+(a \wedge a_n) = b_a$ and $\varphi_n^+(a \wedge a_n') = b \wedge b_a'$ for $a \in M$. This determines a one-to-one homomorphism φ_n^+ on the whole of A_n^+ that extends φ_n.

e. In a similar fashion add b_n to $\varphi_n^+[A_n^+]$ to obtain B_{n+1}; find $a \in A$ so that $\varphi_{n+1} : a \mapsto b_n$ will induce an isomorphism between the algebra A_{n+1}, generated by A_n^+ and a, and the algebra B_{n+1}.

f. The map $\bigcup_n \varphi_n : A \to B$ is an isomorphism.

► **6**. Every countable Boolean algebra is embeddable into $\mathrm{CO}({}^{\omega}2)$. *Hint*: Use 'half' of the previous proof.

2. The space $\mathbb{N}^*$

The space $\mathbb{N}^*$ admits a similar characterization, provided one assumes the Continuum Hypothesis (abbreviated CH). This characterization was proved by Parovichenko in 1963.

2.1. Theorem (CH). *Let X be a compact zero-dimensional space of weight $\mathfrak{c}$, without isolated points, in which nonempty G_δ-sets have nonempty interior and in which disjoint open F_σ-sets have disjoint closures. Then X is homeomorphic to $\mathbb{N}^*$.*

► **1**. The space $\mathbb{N}^*$ has all the properties mentioned in Theorem 2.1.

a. Every nonempty G_δ subset in $\mathbb{N}^*$ has infinite interior. *Hint*: Use Exercise 8.

b. Let U be an open F_σ-set in $\mathbb{N}^*$; there is a countable family of clopen sets whose union is U.

c. If U and V are disjoint open F_σ-sets in $\mathbb{N}^*$ then there is a clopen set C such that $U \subseteq C$ and $C \cap V = \varnothing$. *Hint*: Use Exercise 9.

d. The space $\mathbb{N}^*$ has weight $\mathfrak{c}$. *Hint*: The natural base has cardinality $\mathfrak{c}$. By Exercise 5.d every base must have cardinality $\mathfrak{c}$.

▶ **2**. Let X be compact zero-dimensional and without isolated points.

a. Every nonempty G_δ-set in X has nonempty interior iff $\mathrm{CO}(X)$ satisfies the condition in Exercise 8, i.e., whenever $\langle a_n \rangle_n$ is an increasing sequence in $\mathrm{CO}(X)$ and $a > a_n$ for all n then there is a b such that $a > b > a_n$ for all n.

b. Disjoint open F_σ-sets in X have disjoint closures iff $\mathrm{CO}(X)$ satisfies the condition in Exercise 9, i.e., whenever $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ are sequences in $\mathrm{CO}(X)$ such that $a_m \wedge b_n = \mathbf{0}$ for all m and n there is c such that $a_m \leqslant c$ and $b_n \leqslant c'$ for all m and n.

We first prove half of Parovichenko's theorem.

▶ **3**. Let S be a countable subfamily of $\mathcal{P}(\mathbb{N})/\mathit{fin}$. There is a family $\{a_s : s \in S\}$ in $\mathcal{P}(\mathbb{N})/\mathit{fin}$ such that $\mathbf{0} < a_s \leqslant s$ for all s and $a_s \wedge a_t = \mathbf{0}$ whenever $s \neq t$. *Hint*: Enumerate S as $\{s_n : n \in \omega\}$; using the fact the $\mathcal{P}(\mathbb{N})/\mathit{fin}$ has no atoms construct for every n a family $\{a_{n,i} : i \leqslant n\}$ such that $a_{n,i} \wedge a_{n,j} = \mathbf{0}$ whenever $i \neq j$ and $s_n \geqslant a_{n,n} \geqslant a_{n,n+1} > \cdots$ for all n. Then apply Exercise 8.

▶ **4**. Let A be a countable subalgebra of $\mathcal{P}(\mathbb{N})/\mathit{fin}$. Let L and D be ideals in A such that $a \wedge b = \mathbf{0}$ whenever $a \in L$ and $b \in D$ and put $S = A \setminus (L \cup D)$. Then there is a $c \in \mathcal{P}(\mathbb{N})/\mathit{fin}$ such that $(\forall a \in L)(a \leqslant c)$, $(\forall b \in D)(b \wedge c = \mathbf{0})$ and $(\forall s \in S)\bigl((s \wedge c > \mathbf{0}) \wedge (s \wedge c' > \mathbf{0})\bigr)$.

a. There are increasing and cofinal sequences $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ in L and D respectively.

b. Let $S' = \bigl\{s \in S : (\forall n)(s \nleqslant a_n \vee b_n)\bigr\}$; there is a disjoint refinement $\{d_s : s \in S'\}$ of S' such that $d_s \wedge (a_n \vee b_n) = \mathbf{0}$ for all s and n.

c. For $s \in S'$ choose a_s so that $\mathbf{0} < a_s < d_s$. There is $c \in \mathcal{P}(\mathbb{N})/\mathit{fin}$ such that $a_n \leqslant c$, $a_s = c \wedge d_s$ and $b_n \wedge c = \mathbf{0}$ for all s and n. This c is as required.

▶ **5**. Let A be a subalgebra of B and $x \in B$; the subalgebra generated by A and x consists of all elements of the form

$$(a_1 \wedge x) \vee (a_2 \wedge x') \wedge a_3,$$

where a_1, a_2 and a_3 are from A and pairwise disjoint.

▶ **6**. Let B be a Boolean algebra of cardinality $\aleph_1$. Then B can be embedded into $\mathcal{P}(\mathbb{N})/\mathit{fin}$. Enumerate B as $\{b_\alpha : \alpha < \omega_1\}$ and let B_α be the subalgebra of B generated by $\{b_\beta : \beta < \alpha\}$. Define embeddings $\varphi_\alpha : B_\alpha \to \mathcal{P}(\mathbb{N})/\mathit{fin}$ such that φ_α extends φ_β whenever $\alpha > \beta$.

a. Define $\varphi_0(\mathbf{0}) = \mathbf{0}$ and $\varphi_0(\mathbf{1}) = \mathbf{1}$.

b. If α is a limit ordinal then define $\varphi_\alpha = \bigcup_{\beta<\alpha} \varphi_\beta$.

c. When $\varphi_\alpha : B_\alpha \to \mathcal{P}(\mathbb{N})/\mathit{fin}$ is given and $b_\alpha \notin B_\alpha$ then apply Exercise 4 to find $c \in \mathcal{P}(\mathbb{N})/\mathit{fin}$ such that for all $b \in B_\alpha$ we have $\varphi_\alpha(b) \leqslant c$ iff $b \leqslant b_\alpha$ and $\varphi_\alpha(b) \leqslant c'$ iff $b \leqslant b'_\alpha$.

d. Defining $b_\alpha \mapsto c$ determines an extension $\varphi_{\alpha+1}$ of φ_α. *Hint*: Refer to Exercise 5 and define $\varphi_{\alpha+1}\bigl((a_1 \wedge b_\alpha) \vee (a_2 \wedge b'_\alpha) \vee a_3\bigr) = (\varphi_\alpha(a_1) \wedge c) \vee (\varphi_\alpha(a_2) \wedge c') \vee \varphi_\alpha(a_3)$.

We now state and prove the dual form of Parovichenko's theorem.

2.2. Theorem (CH). *Let A and B be two Boolean algebras of cardinality $\mathfrak{c}$, without atoms and with the properties from Exercises 8 and 9. The A and B are isomorphic.*

► **7**. Prove Theorem 2.2. *Hint*: Adapt Exercise 6.

3. Some examples

We describe a host of examples of spaces that satisfy the conditions of Parovičenko's Theorem. Under CH these spaces are homeomorphic to $\mathbb{N}^*$. These examples all make use of the Čech-Stone compactification.

The Čech-Stone compactification

For convenience we define the Čech-Stone compactification for normal spaces only.

Let X be a normal space and let $\mathcal{C}(X)$ denote its family of closed subsets. Just as on page 4 one defines filters and ultrafilters on $\mathcal{C}(X)$ — Exercise 1.b remains valid (except where it refers to complements). We define βX as the set of ultrafilters on $\mathcal{C}(X)$ and for $A \in \mathcal{C}(X)$ define

$$\overline{A} = \{u \in \beta X : A \in u\}.$$

The family $\bigl\{\overline{A} : A \in \mathcal{C}(X)\bigr\}$ serves as a base for the closed sets of βX.

► **1**. For every $x \in X$ the family $u_x = \{A : x \in A\}$ is an ultrafilter. The map $x \mapsto u_x$ is an embedding of X into βX.

► **2**. a. Let $A, B \in \mathcal{C}(X)$; then $\overline{A \cap B} = \overline{A} \cap \overline{B}$ and $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

b. For $A \in \mathcal{C}(X)$ we have $\operatorname{cl} A = \overline{A}$.

c. The space X is dense in βX.

► **3**. The space βX is compact and Hausdorff.

a. If $\{\overline{A_i} : i \in I\}$ has the finite intersection property then $\{A_i : i \in I\}$ is contained in some ultrafilter.

b. Let u and v be different ultrafilters; find $A \in u$ and $B \in v$ such that $A \notin v$, $B \notin u$ and $A \cup B = X$. Then $\beta X \setminus \overline{A}$ and $\beta X \setminus \overline{B}$ are disjoint neighbourhoods of u and v.

Thus, βX is a compact Hausdorff space that contains X as a dense subset and with the additional property that whenever A and B are closed subsets of X one has $\operatorname{cl}(A \cap B) = \operatorname{cl} A \cap \operatorname{cl} B$.

► **4**. For an open set U in X define $\operatorname{Ex} U = \beta X \setminus \overline{X \setminus U}$. Then $\{\operatorname{Ex} U : U \text{ is open }\}$ is a base for the topology of βX.

σ-compact spaces

We now show that whenever X is σ-compact and locally compact the space $X^* = \beta X \setminus X$ has the two topological properties in Parovičenko's theorem. Whenever A is closed in X we write $A^* = \operatorname{cl} A \cap X^*$.

► **5**. A space X is σ-compact and locally compact iff one can write $X = \bigcup_{n \in \omega} X_n$, where each X_n is compact and $X_n \subseteq \operatorname{int} X_{n+1}$ for all n.

► **6**. Let X be σ-compact and locally compact (but not compact) then every nonempty G_δ-subset of X^* nas nonempty interior — in X^*. Let G be a G_δ-subset of X^* and $u \in G$.
  a. There is an increasing sequence $\langle F_n \rangle_n$ of closed sets in X such that $X^* \setminus G \subseteq \bigcup_n F_n^*$ and $F_n \notin u$ for all n.
  b. For every n there is a k_n such that $X_{k_n} \nsubseteq F_n$.
  c. The set $F = \bigcup_n \operatorname{cl}(F_n \setminus X_{k_n})$ is closed.
  d. For every n we have $\operatorname{cl}(F_n \setminus F) \subseteq X_n$ and hence $F_n^* \cap F^*$.
  e. Choose $x_n \in X_{k_n} \setminus F_n$; the set $B = \{x_n : n \in \omega\}$ is closed and disjoint from F.
  f. In X^* we have $\varnothing \neq B^* \subseteq X^* \setminus F^* \subseteq \operatorname{int} G$.

► **7**. Let X be σ-compact and locally compact (but not compact) then any two disjoint open F_σ-subsets of X^* have disjoint closures — in X^*. Let F and G be disjoint open F_σ-subsets of X^* and consider the subspace $Y = X \cup F \cup G$ of βX.
  a. The space Y is σ-compact, hence Lindelöf, hence normal.
  b. The sets F and G are disjoint and closed in Y. *Hint*: X is locally compact and therefore open in βX.
  c. There are open sets U and V *in* Y with disjoint closures and with $U \supseteq F$ and $V \supseteq G$.
  d. $A = \operatorname{cl}(U \cap X)$ and $B = \operatorname{cl}(V \cap X)$ are closed and disjoint in βX and $F \subseteq A$ and $G \subseteq B$.

Let us call *Parovičenko space* a space that satisfies Parovičenko's conditions, i.e., one that is compact, zero-dimensional, of weight $\mathfrak{c}$, in which nonempty G_δ-sets have nonempty interiors and in which disjoint open F_σ-sets have disjoint closures.

► **8**. The following are all Parovičenko spaces.
  a. $\mathbb{D}^*$, where $\mathbb{D} = \omega \times (\omega + 1)$;
  b. $(\omega \times {}^{\mathbb{N}}2)^*$;
  c. $(\omega \times {}^{\omega_1}2)^*$;

d. $(\omega \times {}^{\mathfrak{c}}2)^*$.

▶ **9**. Let $X = \omega \times {}^{\mathfrak{c}}2$; if $x \in X^*$ then every local base at x has cardinality at least $\mathfrak{c}$. Let $\mathcal{U}$ be a family of fewer than $\mathfrak{c}$ neighbourhoods of x in X^*.

a. If U is clopen in X^* then there is a clopen set C_U in X such that $U = C_U^*$.

b. If C is clopen in X then C is determined by countably many coordinates in $\mathfrak{c}$, i.e., there is a countable subset I of $\mathfrak{c}$ such that $C = \pi_I^{\leftarrow}\bigl[\pi_I[C]\bigr]$, where π_I is the projection of $\omega \times {}^{\mathfrak{c}}2$ onto $\omega \times {}^{I}2$.

c. Choose, for every $U \in \mathcal{U}$, a clopen set C_U in X such that $x \in C_U^* \subseteq U$ and a countable subset I_U such that C_U is determined by the coordinates in I_U.

d. There is $\alpha \in \mathfrak{c} \setminus \bigcup_U I_U$.

e. Let $C = \bigl\{(n,x) \in X : x_\alpha = 0\bigr\}$. Then for all $U \in \mathcal{U}$ we have $C_U^* \nsubseteq C^*$ and $C_U^* \cap C^* \neq \varnothing$.

f. The family $\mathcal{U}$ is not a local base at x.

▶ **10**. Apply Exercise 8 to construct a strictly decreasing sequence $\langle C_\alpha : \alpha < \omega_1\rangle$ of clopen sets in $\mathbb{N}^*$.

a. The family $\mathcal{C} = \bigl\{C \in \mathrm{CO}(\mathbb{N}^*) : (\exists\alpha)\bigl((F_\alpha \subseteq C) \vee (F_\alpha \cap C = \varnothing)\bigr)\bigr\}$ is a Boolean algebra.

b. The Boolean algebra $\mathcal{C}$ satisfies the properties of Exercises 8 and 9.

c. The Stone space $\mathfrak{S}(\mathcal{C})$ of $\mathcal{C}$ is a Parovičenko space.

d. The space $\mathfrak{S}(\mathcal{C})$ is obtained from $\mathbb{N}^*$ by identifying the set $\bigcap_\alpha F_\alpha$ to one point.

e. The space $\mathfrak{S}(\mathcal{C})$ has a point with a local base of cardinality $\aleph_1$.

▶ **11**. The spaces $(\omega \times {}^{\mathfrak{c}}2)^*$ and $\mathfrak{S}(\mathcal{C})$ are Parovičenko spaces that are homeomorphic if and *only if* CH holds.

Chapter 8

Homogeneity

We consider the difference between topological and algebraic homogeneity.

1. Algebraic homogeneity

1.1. Definition. A Boolean algebra B is *homogeneous* if for every nonzero $b \in B$ the Boolean algebra $\{a : a \leqslant b\}$ is isomorphic to B.

► **1**. If X is compact and zero-dimensional then $\mathrm{CO}(X)$ is homogeneous iff X is homeomorphic to each of its nonempty clopen subsets.

► **2**. A Boolean algebra with two (or more) atoms is not homogeneous.

► **3**. The following Boolean algebras are homogeneous.
a. The algebra of clopen subsets of the Cantor set.
b. The algebra $\mathcal{P}(\mathbb{N})/\mathit{fin}$.

► **4**. The measure algebra $\mathbb{M}$ is homogeneous.
a. If B is a Borel set of positive measure then there is an $r \in [0,1]$ such that $m\bigl([0,r] \cap B\bigr) = \frac{1}{2}m(B)$. *Hint*: Consider the function $f : [0,1] \to [0,1]$ defined by $f(r) = m\bigl([0,r] \cap B\bigr)$.
Fix a Borel set B and put $D = \{k2^{-n} : k, n \in \omega, k \leqslant 2^n\}$.
b. There is a set $\{r_d : d \in D\}$ in $[0,1]$ such that, whenever $d < e$ in D we have $m\bigl([d,e] \cap B\bigr) = m(B) \cdot (e - d)$.
Define $f : B \to [0,1]$ by $f(x) = \sup\{d \in D : r_d \leqslant x\}$.
c. For every Borel set A the preimage $f^{\leftarrow}[A]$ is Borel and $m\bigl(f^{\leftarrow}[A]\bigr) = m(B) \cdot m(A)$. *Hint*: The family $\bigl\{A : f^{\leftarrow}[A]$ is Borel and $m\bigl(f^{\leftarrow}[A]\bigr) = m(B) \cdot m(A)\bigr\}$ is a σ-algebra and contains all intervals $[d,e]$ with $d, e \in D$.
d. The map $A \mapsto f^{\leftarrow}[A]$ induces an isomorphism between $\mathbb{M}$ and $\{q(A) : A \subseteq B\}$.

► **5**. The completion of a homogeneous Boolean algebra is also homogeneous.

2. Topological homogeneity

2.1. Definition. A topological space X is *homogeneous* if for any two points x and y of X there is a homeomorphism $h : X \to X$ such that $h(x) = y$.

► **1**. a. The real line $\mathbb{R}$ is homogeneous.
b. The unit interval $[0,1]$ is not homogeneous.
c. The Cantor set ${}^{\omega}2$ is homogeneous. *Hint*: Given x and y let $I = \{n : x_n \neq y_n\}$; define h by flipping the coordinates in I.

3. Interrelations

Although it seems that homogeneous spaces should have homogeneous algebras and vice versa it turns out that neither kind of homogeneity implies the other.

Homogeneous spaces

Most of the homogeneous spaces that one can think of have homogeneous clopen algebras. However there is a homogeneous compact zero-dimensional space X whose clopen algebra $\mathrm{CO}(X)$ is not homogeneous. In [**12**] E. K. van Douwen constructed such an example: it has a measure μ defined on its Borel sets such that if A and B are homeomorphic Borel sets then $\mu(A) = \mu(B)$ and, conversely, whenever clopen sets have the same measure then they are homeomorphic.

Homogeneous algebras

There are many homogeneous algebras whose Stone spaces are not homogeneous. We shall give one specific example and prove a general theorem that provides a potentially large class of such algebras. There is one general positive result however.

▶ **1**. Let B be a homogeneous Boolean algebra such that $\mathfrak{S}(B)$ is first-countable. Then $\mathfrak{S}(B)$ is homogeneous. Let $u, v \in \mathfrak{S}(B)$ and choose countable local bases $\{C_n : n \in \mathbb{N}\}$ and $\{D_n : n \in \mathbb{N}\}$ at u and v respectively, both consisting of clopen sets. Recursively choose autohomeomorphisms h_k of $\mathfrak{S}(B)$, as follows. Put $h_0 = \mathrm{Id}$. If $h_k(u) = v$ or $h_k(v) = u$ then stop; otherwise choose $n_k \geqslant k$ such that $h_k[C_{n_k}]$ is disjoint from D_{n_k} and $h_k[D_{n_k}]$ is disjoint from C_{n_k}; let $f_k : h_k[C_{n_k}] \to D_{n_k}$ be a homeomorphism. Define $g_k : X \to X$ by $g_k(x) = x$ if $x \notin h_k[C_{n_k}] \cup D_{n_k}$, $g_k(x) = f_k(x)$ if $x \in h_k[C_{n_k}]$ and $g_k(x) = f_k^{-1}(x)$ if D_{n_k}. Let $h_{k+1} = g_k \circ h_k$.

a. The map g_k is well-defined and an autohomeomorphism of X.
b. $h_{k+1}[C_{n_k}] = D_{n_k}$ for every k.
c. If $x \notin C_{n_k}$ and $l > k$ then $h_l(x) = h_k(x)$.
d. For every x the limit $h(x) = \lim_k h_k(x)$ exists.
e. The map $h : X \to X$ is a homeomorphism and $h(u) = v$.

$\mathbb{N}^*$ *is not homogeneous*

We shall see that $\mathbb{N}^*$ is not homogeneous, even though the Boolean algebra $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is. The first result is due to W. Rudin.

▶ **2**. The Continuum Hypothesis implies that $\mathbb{N}^*$ is not homogeneous.

a. Apply Exercise 10 and Theorem 2.1 to deduce that $\mathbb{N}^*$ has a P-point u, i.e., $u \in \operatorname{int} \bigcap \mathcal{U}$, whenever $\mathcal{U}$ is a countable family of neighbourhoods of u.
b. Let C be any countably infinite subset of $\mathbb{N}^*$ and let v be an accumulation point of C, then v is not a P-point.

c. There is no autohomeomorphism of $\mathbb{N}^*$ that maps u to v.

The next result is due to Frolík. It shows that CH is not necessary to prove that $\mathbb{N}^*$ is not homogeneous.

▶ **3**. Let X be a regular space and $D = \{d_n : n \in \mathbb{N}\}$ a relatively discrete subset of X; then there is a *disjoint* family $\{U_n : n \in \mathbb{N}\}$ of open sets with $d_n \in U_n$ for all n. *Hint*: Choose the U_n recursively and with the property that $\operatorname{cl} U_n \cap D = \{d_n\}$ and $\operatorname{cl} U_n \cap \operatorname{cl} U_m = \varnothing$ whenever $m < n$.

▶ **4**. Let $f : \mathbb{N} \to \mathbb{N}$ be a map and $A = \{n : f(n) \neq n\}$. One can partition A into three sets A_0, A_1 and A_2 such that $f[A_i]$ is always disjoint from A_i. For $n \in A$ let $A_n = \{f^k(n) : k \in \mathbb{N}\} \cap A$ and define a relation $m \leftrightsquigarrow n$ on A by $A_n \cap A_m \neq \varnothing$.

a. The relation $\leftrightsquigarrow$ is an equivalence relation on A.

Let E be an equivalence class.

b. Case 1: there is an $n \in E$ such that $f(n) \notin A$ then n is unique (in E) and for every $m \in E$ there is a k_m such that $f^{k_m}(m) = n$; $E_0 = \{m \in E : k_m \text{ is even}\}$ and $E_1 = \{m : k_m \text{ is odd}\}$ both satisfy $f[E_i] \cap E_i = \varnothing$.

c. Case 2: there are an $n \in E$ and a (minimal) non-zero k such that $f^k(n) = n$. For every $m \in E$ there is a $f^k(m) = n$, let k_m be the minimal such k; $E_0 = \{m \in E : k_m \text{ is even}\}$ and $E_1 = \{m : k_m \text{ is odd}\}$ both satisfy $f[E_i] \cap E_i = \varnothing$, *except* when k is odd, then we delete n from E_0 and put $E_2 = \{n\}$.

d. Case 3 (the remaining case): for every $m, n \in E$ there are minimal $k_{m,n}$ and $l_{m,n}$ such that $f^{k_{m,n}}(m) = f^{l_{m,n}}(n)$. Fix $n \in E$ and define $E_0 = \{m \in E : k_{m,n} + l_{m,n} \text{ is odd}\}$ and $E_1 = \{m \in E : k_{m,n} + l_{m,n} \text{ is even}\}$; then $f[E_i] \cap E_i = \varnothing$ for $i = 0$ and 1.

▶ **5**. Let $u \in \mathbb{N}^*$ and let $D = \{d_n : n \in \mathbb{N}\}$ be a relatively discrete subset of $\mathbb{N}^*$. There is no homeomorphism $h : \mathbb{N}^* \to \mathbb{N}^*$ such that $h(d_u) = u$. Assume there is such a homeomorphism and write $e_n = h(d_n)$.

a. $E = \{e_n : n \in \mathbb{N}\}$ is relatively discrete and $u = e_u$.

b. There are disjoint subsets B_n of $\mathbb{N}$ such that $B_n \in e_n$ for all n and $\mathbb{N} = \bigcup_n B_n$.

c. For a subset A of $\mathbb{N}$ we have $A \in u$ iff $\bigcup_{n \in A} B_n \in u$. *Hint*: $A \in u$ iff $u \in \operatorname{cl}\{e_n : n \in A\}$.

d. Define $f : \mathbb{N} \to \mathbb{N}$ by $f(i) = n$ iff $i \in B_n$. Then $f(u) = u$.

e. The set $\{n : f(n) \neq n\}$ does not belong to u. *Hint*: Apply Exercise 4.

f. Consider $A = \{n : f(n) = n\}$. For every $n \in A$ we have $A \cap B_n = \{n\}$; this gives rise to a contradiction, to wit that $e_n = n$ for all $n \in A$.

The following theorem provides us with a large supply of nonhomogeneous spaces.

3.1. THEOREM (FROLÍK). *No infinite compact F-space is homogeneous.* □

An F-space is one in which disjoint open F_σ-sets have disjoint closures.

▶ **6**. Let X be a compact F-space and let A and B be *separated* countable subsets of X, i.e., $A \cap \operatorname{cl} B = \varnothing = \operatorname{cl} A \cap B$. Then $\operatorname{cl} A$ and $\operatorname{cl} B$ are disjoint. *Hint*: Enumerate

A and B as $\{a_n : n \in \mathbb{N}\}$ and $\{b_n : n \in \mathbb{N}\}$ respectively. Recursively choose open F_σ-sets U_n around a_n and V_n around b_n such that $\operatorname{cl} U_n \cap (\operatorname{cl} B \cup \bigcup_{m<n} \operatorname{cl} V_m) = \varnothing$ and $\operatorname{cl} V_n \cap (\operatorname{cl} A \cup \bigcup_{m\leqslant n} \operatorname{cl} U_m) = \varnothing$. Consider $\bigcup_n U_n$ and $\bigcup_n V_n$.

► **7**. Let $D = \{d_n : n \in \mathbb{N}\}$ be a countable and relatively discrete subset of a compact F-space X. The map $f : n \mapsto d_n$ induces a homeomorphism from $\beta\mathbb{N}$ onto D.

a. If A and B are disjoint subsets of $\mathbb{N}$ then $\operatorname{cl} f[A]$ and $\operatorname{cl} f[B]$ are disjoint.

b. If $u \in \beta\mathbb{N}$ then $\bigcap_{A\in u} \operatorname{cl} f[A]$ consists of one point d_u.

c. The map $g : u \mapsto d_u$ is one-to-one and continuous. *Hint*: If V is an open set in X and $A = \{n : d_n \in V\}$ then $\operatorname{cl} f[A] = \operatorname{cl} V$.

► **8**. Let X be an infinite compact F-space and take a countable and discrete subset of it, which we identify with $\mathbb{N}$.

a. The closure of $\mathbb{N}$ is homeomorphic with $\beta\mathbb{N}$.

b. There are neighbourhoods U_n of n such that $\operatorname{cl} U_m \cap \operatorname{cl} U_n = \varnothing$ whenever $m \neq n$ and $\operatorname{cl} U_n \cap \operatorname{cl} \mathbb{N} = \{n\}$ for all n.

Let u and v be points in $\mathbb{N}^*$ and assume $h : X \to X$ is a homeomorphism such that $h(u) = v$. For $n \in \mathbb{N}$ write $d_n = h(n)$ and $V_n = h[U_n]$. Put $A = \{n : d_n \in \bigcup_m \operatorname{cl} U_m\}$, $B = \{n : d_n \in \mathbb{N}^*\}$ and $C = \mathbb{N} \setminus (A \cup B)$.

c. If $U \subseteq \mathbb{N}$ then $U \in u$ iff $u \in \operatorname{cl} \bigcup_{m\in U} U_m$.

d. If $U \subseteq \mathbb{N}$ then $U \in u$ iff $v \in \operatorname{cl}\{d_n : n \in U\}$.

e. $\operatorname{cl} N \cap \operatorname{cl}\{n : n \in C\} = \varnothing$, hence $C \notin u$. *Hint*: $\mathbb{N} \cap \operatorname{cl}\{d_n : n \in C\} = \varnothing = \operatorname{cl} N \cap \{d_n : n \in C\}$.

f. If $A \in u$ then define $f : A \to \mathbb{N}$ by $f(n) = m$ iff $d_n \in \operatorname{cl} U_m$. Then $f(u) = v$. *Hint*: If $U \in u$ then $v \in \operatorname{cl}\{d_n : n \in U\}$ and $\operatorname{cl}\{d_n : n \in U\} \subseteq \operatorname{cl} \bigcup_{m\in f[U]} U_m$.

g. If $B \in u$ then define $f : \mathbb{N} \to B$ by $f(n) = m$ iff $n \in \operatorname{cl} V_m$. Then $f(v) = u$. *Hint*: If $V \in v$ then $v \in \operatorname{cl}_{m\in f[V]} \operatorname{cl} V_m$.

We see that in order to show that no infinite compact F-space is homogeneous we must produce two ultrafilters u and v on $\mathbb{N}$ such that there is *no* mapping $f : \mathbb{N} \to \mathbb{N}$ with $f(u) = v$ or $f(v) = u$.

For the construction of such ultrafilters we shall need the independent family $\{I_x : x \in \mathcal{P}(\mathbb{N})\}$, constructed in Exercise 2. We introduce two pieces of notation: for $x \in \mathcal{P}(\mathbb{N})$ we write $I_{x,1} = I_x$ and $I_{x,0} = \mathbb{N} \setminus I_x$. Also, for any set S we let $\operatorname{Fn}(S, 2)$ denote the set of functions p whose domain is a finite subset of S and whose range is contained in 2. Observe that the independence of $\{I_x : x \in \mathcal{P}(\mathbb{N})\}$ can be expressed as follows: for every $p \in \operatorname{Fn}(\mathcal{P}(\mathbb{N}), 2)$ the set $\bigcap_{x\in\operatorname{dom} p} I_{x,p(x)}$ is infinite. Finally, if $S \subseteq \mathcal{P}(\mathbb{N})$ and if $\mathcal{F}$ is a filter then we say that $\{\mathbb{I}_x : x \in S\}$ is *independent with respect to* $\mathcal{F}$ if $F \cap \bigcap x \in \operatorname{dom} p I_{x,p(x)} \neq \varnothing$ whenever $F \in \mathcal{F}$ and $p \in \operatorname{Fn}(S, 2)$.

► **9**. Assume $\{I_x : x \in S\}$ is independent with respect to $\mathcal{F}$ and that $A \in \mathcal{P}(\mathbb{N})$. Then *either* $\{I_x : x \in S\}$ is independent with respect to the filter generated by $\mathcal{F}$ and A *or* there is a finite subset s of S such that $\{I_x : x \in S \setminus s\}$ independent with respect to the filter generated by $\mathcal{F}$ and A^c. *Hint*: Write out what the negation of

"$\{I_x : x \in S\}$ is independent with respect to the filter generated by $\mathcal{F}$ and A" entails — and consider the possibility that $\mathcal{F}$ and A do not generate a filter.

▶ **10**. Assume $\{I_x : x \in S\}$ is independent with respect to the filters $\mathcal{F}$ and $\mathcal{G}$, let $f : \mathbb{N} \to \mathbb{N}$ be any map and let $x \in S$. One can choose $i \in \{0,1\}$ and a finite subset s of S such that 1) $\mathcal{G}$ and $f^{\leftarrow}[I_{x,i}]$ generate a filter $\mathcal{G}'$; 2) $\{I_x : x \in S \setminus s\}$ is independent with respect to $\mathcal{G}'$; and 3) $x \in s$. *Hint*: Apply the previous exercise and observe that enlarging s will not spoil independence.

▶ **11** (KUNEN [**5**]). There are ultrafilters u and v on $\mathbb{N}$ such that there is *no* map $f : \mathbb{N} \to \mathbb{N}$ with $f(u) = v$ or $f(v) = u$. Let $\{I_x : x \in \mathcal{P}(\mathbb{N})\}$ be an independent family; let $\{A_\alpha : \alpha < \mathfrak{c}\}$ enumerate $\mathcal{P}(\mathbb{N})$ and let $\{f_\alpha : \alpha < \mathfrak{c}\}$ enumerate ${}^{\mathbb{N}}\mathbb{N}$. Recursively build filters $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ and subsets S_α of $\mathcal{P}(\mathbb{N})$ such that for every α the family $\{I_x : x \in S_\alpha\}$ is independent with respect to $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$. Start with $\mathcal{F}_0 = \mathcal{G}_0 = \{A : A^c \text{ is finite}\}$ and $S_0 = \mathcal{P}(\mathbb{N})$. Let $\alpha < \mathfrak{c}$.

a. There are a finite subset s of S_α and a filter $\mathcal{F}'_\alpha$ that extends $\mathcal{F}_\alpha$ such that A_α or A^c_α belongs to $\mathcal{F}'_\alpha$ and $\{I_x : x \in S_\alpha \setminus s\}$ is independent with respect to $\mathcal{F}'_\alpha$.

b. There are a finite subset s' of $S_\alpha \setminus s$ and a filter $\mathcal{G}'_\alpha$ that extends $\mathcal{G}_\alpha$ such that A_α or A^c_α belongs to $\mathcal{G}'_\alpha$ and $\{I_x : x \in S_\alpha \setminus (s \cup s')\}$ is independent with respect to $\mathcal{G}'_\alpha$.

c. There are a finite subset s'' of $S_\alpha \setminus (s \cup s')$ an $x_\alpha \in s''$ and an $i_\alpha \in \{0,1\}$ such that $\mathcal{G}'_\alpha$ and $f^{\leftarrow}_\alpha[I_{x_\alpha,i_\alpha}]$ generate a filter $\mathcal{G}''_\alpha$ and $\{I_x : x \in S_\alpha \setminus (s \cup s' \cup s'')\}$ is independent with respect to $\mathcal{G}''_\alpha$.

d. $\mathcal{F}'_\alpha$ and $I_{x_\alpha,1-i_\alpha}$ generate a filter $\mathcal{F}''_\alpha$.

e. There are a finite subset s''' of $S_\alpha \setminus (s \cup s' \cup s'')$ a $y_\alpha \in s'''$ and a $j_\alpha \in \{0,1\}$ such that $\mathcal{F}''_\alpha$ and $f^{\leftarrow}[I_{y_\alpha,j_\alpha}]$ generate a filter $\mathcal{F}'''_\alpha$ and $\{I_x : x \in S_\alpha \setminus (s \cup s' \cup s'' \cup s''')\}$ is independent with respect to $\mathcal{F}'''_\alpha$.

f. $\mathcal{G}''_\alpha$ and $I_{y_\alpha,1-j_\alpha}$ generate a filter $\mathcal{G}'''_\alpha$.

g. Setting $\mathcal{F}_{\alpha+1} = \mathcal{F}'''_\alpha$, $\mathcal{G}_{\alpha+1} = \mathcal{G}'''_\alpha$ and $S_{\alpha+1} = S_\alpha \setminus (s \cup s' \cup s'' \cup s''')$ keeps the conditions satisfied and has $S_\alpha \setminus S_{\alpha+1}$ finite.

If α is a limit ordinal set $\mathcal{F}_\alpha = \bigcup_{\beta<\alpha} \mathcal{F}_\beta$, $\mathcal{G}_\alpha = \bigcup_{\beta<\alpha} \mathcal{G}_\beta$ and $S_\alpha = \bigcap_{\beta<\alpha} S_\beta$.

h. For every α the cardinality of $\mathcal{P}(\mathbb{N}) \setminus S_\alpha$ is less than $\mathfrak{c}$; in particular S_α is never empty.

Let $u = \bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha$ and $v = \bigcup_{\alpha<\mathfrak{c}} \mathcal{G}_\alpha$.

i. u and v are ultrafilters.

j. $f_\alpha(u) \neq v$ and $f(v) \neq u$ for every α. *Hint*: $I_{x_\alpha,1-i_\alpha} \in u$ and $f^{\leftarrow}[I_{x_\alpha,i_\alpha}] \in v$.

This theorem and its proof caused a lot of research. The point is that, although it provides two points such that no homeomorphism maps one to the other, one cannot see by just looking at the points that there is no such homeomorphism. One of the first results is Kunen's theorem from [**6**] that in $\mathbb{N}^*$ there is a point u with the following property: if $C \subseteq \mathbb{N}^*$ is countable and $u \notin C$ then $u \notin \operatorname{cl} C$. Since there are also points that do not have this property this provides points with clearly different properties.

Appendix A

The Axioms of Set Theory

In the parlance of Mathematical Logic, Set Theory is a first-order theory with equality and one binary predicate, denoted $\in$, with the following axioms.

The Axiom of Extensionality. Sets with the same elements are equal: $(\forall x)(x \in a \leftrightarrow x \in b) \to (a = b)$.

The Axiom of Pairing. For any two sets a and b there is a third set having only a and b as its elements: $(\forall a)(\forall b)(\exists c)(\forall x)\bigl(x \in c \leftrightarrow (x = a \vee x = b)\bigr)$.

The Axiom of Union. For any set a there is a set consisting of all the elements of the elements of a: $(\forall a)(\exists b)(\forall x)\bigl(x \in b \leftrightarrow (\exists y)(y \in a \wedge x \in y)\bigr)$.

The Axiom of Power Set. For any set a there is a set consisting of all the subsets of a: $(\forall a)(\exists b)(\forall x)\bigl(x \in b \leftrightarrow (\forall y)(y \in x \to y \in a)\bigr)$.

The Axiom of Separation. If φ is a property, possibly with a parameter p, then for every a and p there is a set that consists of those elements of a that satisfy φ: $(\forall a)(\forall p)(\exists b)(\forall x)(x \in b \leftrightarrow (x \in a \wedge \varphi(x,p)))$.

The Axiom of Replacement. If F is a function then for every set a its image $F[a]$ under F is a set: $(\forall a)(\exists b)(\forall y)(y \in b \leftrightarrow (\exists x)(x \in a \wedge F(x) = y))$.

The Axiom of Infinity. There is an infinite set: $(\exists a)(\varnothing \in a \wedge (\forall x)(x \in a \to x \cup \{x\} \in a)$.

The Axiom of Foundation. Every nonempty set has a $\in$-minimal element: $(\forall a)\bigl(a \neq \varnothing \to (\exists b)(b \in a \wedge (\forall c)(c \in b \to c \notin a))\bigr)$.

The Axiom of Choice. Every set of nonempty sets has a choice function: $(\forall a)(\exists b)\bigl((\forall x \in a)(\exists y \in x)(\langle x,y\rangle \in b) \wedge (\forall x)(\forall y)(\forall z)((\langle x,y\rangle \in b \wedge \langle x,z\rangle \in b) \to y = z)\bigr)$.

These axioms form the starting point for Set Theory, just like Euclid's axioms were the starting point for Euclidean geometry.

The Axiom of Extensionality connects $=$ and $\in$; it mirrors the way in which we normally show that sets are equal. The Axiom of Pairing, combined with the Axiom of Extensionality, lets us define a new 'function': $\{a,b\}$ is the *unique* c such that $(\forall x)(x \in c \leftrightarrow (x = a \vee x = b))$. We can then form $\{a\} = \{a,a\}$, the singleton set, and $\bigl\{\{a\},\{a,b\}\bigr\}$, the *ordered* pair, usually denoted $\langle a,b\rangle$.

▶ **12**. Verify that $\langle a,b\rangle = \langle c,d\rangle$ iff $a=c$ and $b=d$.

The Axioms of Union and Power Set give additional operations: $\bigcup a = \bigl\{y : (\exists x \in a)(y \in x)\bigr\}$ and $\mathcal{P}(a) = \bigl\{x : (\forall y \in x)(y \in a)\bigr\}$. By combining Union and Pairing we can form $a \cup b = \bigcup\{a,b\}$.

The Axioms of Separation and Replacement deserve special consideration; both in fact represent an infinite list of axioms, one for each property or function. As such they should properly called axiom schemas. As an application let $\varphi(x,p)$ be $x \in p$; then for any sets a and p the set $\{x \in a : x \in p\}$ exists — it is of course nothing but $a \cap p$. Similarly, if $\varphi(x,p)$ is $(\forall y \in p)(x \in p)$ and $a \in p$ then $\{x \in a : \varphi(x,p)\}$ defines $\bigcap p$.

In the Replacement schema proper one considers formulas that define functions: if $\varphi(x,y,p)$ satisfies $(\forall x)(\forall y)(\forall z)(\varphi(x,y,p) \land \varphi(x,z,p) \to y = z)$ then $(\forall a)(\exists b)(\forall y)\bigl(y \in b \leftrightarrow (\exists x \in a)\varphi(x,y,p)\bigr)$.

The Axiom of Infinity may look strange at first but we must realize that none of the axioms so far can express the notions of 'finite' and 'infinite' in any way. As can be seen in Appendix B the present formulation leads to a satisfactory *set* that does all we may expect of the natural numbers. A set as in the Axiom of Infinity is called *inductive* and an inductive set deserves to be called infinite because it contains the chain $\varnothing \in \{\varnothing\} \in \bigl\{\varnothing, \{\varnothing\}\bigr\} \in \cdots$ that goes on forever. The Axiom of Foundation is also called the Axiom of Regularity because it proscribes infinite chains $\cdots \in x_2 \in x_1 \in x_0$ and thus ensures that the universe of sets can be built up by iterating the power set operation, thus: $V_0 = \varnothing$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ and $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ if α is a limit ordinal.

▶ **13**. Every set belongs to V_α for some α. *Hint*: Given a let $b = \bigl\{x \in \operatorname{trcl} a : (\exists\alpha)(x \in V_\alpha)\bigr\}$. If $b \neq \operatorname{trcl} a$ let x be $\in$-minimal in $\operatorname{trcl} a \setminus b$; by the Axiom of Replacement $x \subseteq V_\alpha$ for some α and so $x \in V_{\alpha+1}$ and hence $x \in b$.

The Axiom of Choice accounts for the C in ZFC. Because of its nonconstructive nature — the existence of the choice function is simply asserted, no description is given — it is treated with suspicion by some. We will use it freely in this book and at some places point out some of its stranger consequences.

Appendix B

Basics of Set Theory

In this chapter we collect some notions from Set Theory that are used throughout the book. We take the opportunity to illustrate how familiar set-theoretic operations can be justified on the basis of the axioms presented in Chapter A.

1. The Natural numbers

To see how $\mathbb{N}$ can be conceived as a set we apply the Axiom of Infinity to get an inductive set I. This means that $\varnothing \in I$ and that $x \cup \{x\} \in I$ whenever $x \in I$.

▶ **1**. There is a smallest inductive set.
  a. Apply the Power Set and Separation Axioms to construct $\mathbb{N} = \bigcap\{X : X \subseteq I$ and X is inductive$\}$.
  b. The set $\mathbb{N}$ is inductive and a subset of any other inductive set.

Thus, the official Set-Theoretic definition of $\mathbb{N}$ is that it is the smallest inductive set. We make some abbreviations: $0 = \varnothing$, $1 = \{0\}$, $2 = \{0, 1\}$, and so on.

▶ **2**. The set $\mathbb{N}$, together with the operation $n \mapsto n \cup \{n\}$, satisfies Peano's Axioms for the natural numbers.

This exercise allows us to define addition and multiplication as usual and $m < n$ by $(\exists k)(k \neq 0 \wedge n = m + k)$.

▶ **3**. The order $<$ is identical to $\in$.

2. Products and relations

Relations abound in mathematics; they have a reasonably simple mathematical foundation.

Products

Given two sets X and Y we define $X \times Y$ to be the set of ordered pairs $\langle x, y\rangle$ with $x \in X$ and $y \in Y$.

► **1**. The existence of $X \times Y$ can be deduced from the Axioms of Pairing, Union, Power Set and Separation.

a. If $x \in X$ and $y \in Y$ then $\langle x, y\rangle \in \mathcal{P}\big(\mathcal{P}(X \cup Y)\big)$.

b. $X \times Y = \big\{z \in \mathcal{P}\big(\mathcal{P}(X \cup Y)\big) : (\exists x \in X)(\exists y \in Y)(z = \langle x, y\rangle)\big\}$.

There will be situations where the Power Set Axiom is not available; we can avoid it in building $X \times Y$.

► **2**. The existence of $X \times Y$ can be deduced from the Axioms of Pairing, Union, and Replacement.

a. Given $x \in X$ use the map $y \mapsto \langle x, y\rangle$ to deduce that $\{x\} \times Y$ is a set.

b. Use the map $x \mapsto \{x\} \times Y$ to deduce that $\mathcal{X} = \big\{\{x\} \times Y : x \in X\big\}$ is a set.

c. $X \times Y = \bigcup \mathcal{X}$ is a set.

Relations

A relation is a set of ordered pairs. Its domain is the set of its first coordinates and its range the set of its second coordinates.

► **3**. a. z is an ordered pair iff $(\exists u \in z)(\exists v \in z)(\exists x \in u)(\exists y \in v)\big(z = \langle x, y\rangle\big)$.

b. x is the first coordinate of z iff $(\exists v \in z)(\exists y \in v)\big(z = \langle x, y\rangle\big)$.

c. y is the second coordinate of z iff $(\exists u \in z)(\exists x \in u)\big(z = \langle x, y\rangle\big)$.

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