Chapter 3

Arkhangel′skiĭ's theorem

A special case of the theorem of the title says that first-countable compact Hausdorff spaces have cardinality at most $\mathfrak{c}$. In the literature one can find three approaches to this result; we shall present each of these, in an attempt to show how better tools do make for lighter work. For expository purposes we confine ourselves to the basic case of first-countable compact Hausdorff spaces; at the end of this section we indicate possible generalizations.

1. First proof

This is essentially Arkhangel′skiĭ's original proof. We shall require a few preliminary topological results.

▶ **1**. Let X be first-countable Hausdorff space with a dense set of cardinality $\mathfrak{c}$ (or less); then $|X| \leqslant \mathfrak{c}$. *Hint*: Every point in the space is the limit of a sequence from the dense set.

▶ **2**. Let X be a first-countable compact Hausdorff space and A a closed subset of cardinality $\mathfrak{c}$ (or less); then $X \setminus A$ can be written as the union of no more than $\mathfrak{c}$ closed sets. *Hint*: Choose a countable local base $\mathcal{B}_x$ at each point x of A and consider the family of all finite covers of A whose members belong to $\bigcup_{x\in A} \mathcal{B}_x$.

1.1. Theorem. *Let X be a first-countable compact Hausdorff space; then $|X| \leqslant \mathfrak{c}$.*

▶ **3**. Prove Theorem 1.1. Let T denote the tree ${}^{<\omega_1}\mathfrak{c}$ of countable sequences of elements of $\mathfrak{c}$.

a. $|T| = \mathfrak{c}$.

Choose closed sets F_t, for all $t \in T$, and points x_t, for $t \in T$ of successor height, as follows. First, $F_\varnothing = X$ and $x_\varnothing$ is any point of X. Second, if $\operatorname{ht} t$ is a limit ordinal we let $F_t = \bigcap_{s<t} F_s$. Third, we define $F_{t,\alpha}$ and $x_{t,\alpha}$ for every $\alpha < \mathfrak{c}$: Let $A_t = \operatorname{cl}\{x_s : s \leqslant t\}$ and write $X \setminus A_t = \bigcup_{\alpha<\mathfrak{c}} G_{t,\alpha}$, where each $G_{t,\alpha}$ is closed. Now put $F_{t,\alpha} = F_t \cap G_{t,\alpha}$ and let $x_{t,\alpha}$ be any point of $F_{t,\alpha}$ unless this set is empty, in which case we let $x_{t,\alpha} = x_\varnothing$.

b. $F_t \subseteq A_t \cup \bigcup_{\alpha<\mathfrak{c}} F_{t,\alpha}$.

c. For every α we have $X = \bigcup\{A_t : \operatorname{ht} t = \alpha\} \cup \bigcup\{F_t : \operatorname{ht} t = \alpha\}$.

Let $T' = \{t : |F_t| \leqslant \mathfrak{c}\}$.

d. $\bigcup_{t\in T} A_t \cup \bigcup_{t\in T'} F_t$ has cardinality $\mathfrak{c}$ (or less).

Assume $X \neq \bigcup_{t\in T} A_t \cup \bigcup_{t\in T'} F_t$ and choose $x \in X$ outside the union.

e. There is a path P through T such that $x \in F_t$ for all $t \in P$.
f. $\mathrm{cl}\{x_s : s < t\} \cap \mathrm{cl}\{x_s : t \leqslant s, s \in P\} = \varnothing$, whenever $t \in P$.
g. If $y \in \mathrm{cl}\{x_s : s \in P\}$ then $y \in \mathrm{cl}\{x_s : s < t\}$ for some $t \in P$; therefore $\bigcap_{t \in P} \mathrm{cl}\{x_s : t \leqslant s, s \in P\} = \varnothing$.
h. X is compact, hence $\bigcap_{t \in P} \mathrm{cl}\{x_s : t \leqslant s, s \in P\} \neq \varnothing$.

2. Second proof

The first proof is tree-like; the second proof proceeds in a linear recursion.

► **1**. Prove Theorem 1.1. Fix for every $x \in X$ a countable local base $\mathcal{B}_x$. Recursively define closed sets F_α, for $\alpha \in \omega_1$, as follows. $F_0 = \{x_0\}$ for some x_0. If α is a limit ordinal let $F_\alpha = \mathrm{cl} \bigcup_{\beta < \alpha} F_\beta$. If F_α is given let $\mathcal{B}_\alpha = \bigcup_{x \in F_\alpha} \mathcal{B}_x$ and choose for every finite subfamily $\mathcal{U}$ of $\mathcal{B}_\alpha$ that covers F_α but not X one point $x_{\mathcal{U}} \in X \setminus \bigcup \mathcal{U}$ and let $F_{\alpha+1}$ be the closure of the union of F_α and the set of all points $x_{\mathcal{U}}$.
a. For every α we have $|F_\alpha| \leqslant \mathfrak{c}$ and $|\mathcal{B}_\alpha| \leqslant \mathfrak{c}$.
b. The set $F = \bigcup_\alpha F_\alpha$ is closed, hence compact.
Let $\mathcal{U}$ be a finite subfamily of $\bigcup_{x \in F} \mathcal{B}_x$ that covers F.
c. $\mathcal{U} \subseteq \mathcal{B}_\alpha$ for some α.
d. $\mathcal{U}$ covers X. *Hint*: $\mathcal{U}$ covers $F_{\alpha+1}$.
e. Deduce that $X = F$, hence $|X| \leqslant \mathfrak{c}$.

3. Third proof

The third proof is the second proof in disguise.

► **1**. Prove Theorem 1.1. Fix for every $x \in X$ a countable local base $\mathcal{B}_x$. Let θ be large enough so that X and the assignment $x \mapsto \mathcal{B}_x$ belong to $H(\theta)$. Take an elementary substructure M of $H(\theta)$, of cardinality $\mathfrak{c}$, and such that X and $x \mapsto \mathcal{B}_x$ belong to M and ${}^\omega M \subseteq M$.
a. $F = X \cap M$ is closed in X. *Hint*: If $x \in \mathrm{cl}(X \cap M)$ then some sequence in $X \cap M$ converges to x; the sequence belongs to M.
b. Every finite subfamily $\mathcal{U}$ of $\bigcup_{x \in F} \mathcal{B}_x$ belongs to M; if it covers F then it also covers X. *Hint*: $M \vDash (\forall x \in X)(\exists U \in \mathcal{U})(x \in U)$.

4. Extensions and generalizations

One can relax the assumptions of Theorem 1.1 considerably.

► **1**. Theorem 1.1 also holds for Lindelöf spaces. *Hint*: All the proofs go through with finite collections replaced by countable ones.

We can replace the assumption of first-countability by the conjunction of two weaker properties: *countable pseudocharacter*, i.e., points are G_δ-sets, and *countable tightness*, which means that whenever $x \in \operatorname{cl} A$ there is a countable subset B of A such that $x \in \operatorname{cl} B$.

First we rework Exercise 1.1.

► **2**. Let X be a Lindelöf space with countable pseudocharacter and countable tightness. If A is a subset of X of cardinality $\mathfrak{c}$ or less then also $|\operatorname{cl} A| \leqslant \mathfrak{c}$.

a. It suffices to show that $|\operatorname{cl} A| \leqslant \mathfrak{c}$ whenever A is countable. *Hint*: $\operatorname{cl} A = \bigcup\{\operatorname{cl} B : B \in [A]^{\leqslant \aleph_0}\}$.

Assume X itself is separable and let D be a countable dense subset.

b. For every x we have $\{x\} = \bigcap\{O : x \in O$ and O is regular open$\}$.

c. X has at most $\mathfrak{c}$ regular open sets. *Hint*: If O is regular open then $O = \operatorname{int} \operatorname{cl}(O \cap D)$.

For every countable family $\mathcal{U}$ of regular open sets put $N_{\mathcal{U}} = X \setminus \bigcup \mathcal{U}$ and let $\mathcal{N}$ be the family of these $N_{\mathcal{U}}$'s.

d. If O is open and $x \in O$ then there is a $\mathcal{U}$ such that $x \in N_{\mathcal{U}} \subseteq O$. *Hint*: $X \setminus O$ is Lindelöf.

e. For every point x there is a countable subfamily $\mathcal{N}_y$ of $\mathcal{N}$ such that $\{x\} = \bigcap \mathcal{N}_y$.

f. The map $x \mapsto \mathcal{N}_y$ from X into $[\mathcal{N}]^{\leqslant \aleph_0}$ is one-to-one.

Exercise 1.2 needs less extra work.

► **3**. Let X be a Lindelöf space of countable pseudocharacter and A a closed subset of cardinality $\mathfrak{c}$ (or less); then $X \setminus A$ can be written as the union of no more than $\mathfrak{c}$ closed sets. *Hint*: Choose a countable family $\mathcal{B}_x$ of open sets at each point x of A with $\bigcap \mathcal{B}_x = \{x\}$ and consider the family of all countable covers of A whose members belong to $\bigcup_{x \in A} \mathcal{B}_x$.

► **4**. Use any of the three proofs to show that a Lindelöf Hausdorff space of countable pseudocharacter and countable tightness has cardinality at most $\mathfrak{c}$.