Chapter 5

Balogh's Dowker space

Balogh's example is constructed using pairs of elementary substructures of the universe. To see how it works we look at an easier example first.

1. An example of Rudin's

We discuss an example of a normal space that is not collectionwise Hausdorff, it is an adaptation of an example due to Rudin.

The space will have $\mathfrak{c} \cup [\mathfrak{c}]^2$ as its underlying set, where $[\mathfrak{c}]^2$ denotes $\bigl\{\{\alpha,\beta\} : \alpha < \beta < \mathfrak{c}\bigr\}$. Each of the points $\{\alpha,\beta\}$ will be isolated. For each α we will find a filter $\mathcal{F}_\alpha$ of subsets of $\mathfrak{c}$ and define the neighbourhoods of α to be the sets of the form $U(\alpha,F) = \{\alpha\} \cup \bigl\{\{\alpha,\beta\} : \beta \in F\bigr\}$, with $F \in \mathcal{F}_\alpha$.

▶ **1.** a. $U(\alpha,F) \cap U(\beta,G) \subseteq \bigl\{\{\alpha,\beta\}\bigr\}$.
b. $U(\alpha,F) \cap U(\alpha,G) \neq \varnothing$ iff $\alpha \in G$ and $\beta \in F$.

We shall choose for every $\alpha \in \mathfrak{c}$ and every subset A of $\mathfrak{c}$ a subset $F(\alpha,A)$ and let $\mathcal{F}_\alpha$ be the filter generated by $\{F(\alpha,A) : A \subseteq \mathfrak{c}\}$. Note that $\mathcal{F}_\alpha$ may be an improper filter.

Normality will be achieve by ensuring that $\beta \notin F(\alpha,A)$ or $\alpha \notin F(\beta,A)$ whenever $\alpha \notin A$ and $\beta \in A$.

To this end we define $I(\alpha,A) = A$ if $\alpha \in A$ and $I(\alpha,A) = \mathfrak{c} \setminus A$ if $\alpha \notin A$. We shall also define sets $J(\alpha,A)$ for all α and A and put

$$F(\alpha,A) = I(\alpha,A) \cup \{\beta > \alpha : \beta \in J(\alpha,A)\} \cup \{\beta < \alpha : \alpha \notin J(\beta,A)\}.$$

This already gives us normality.

▶ **2**. If $A \subseteq \mathfrak{c}$, $\alpha \in A$ and $\beta \notin A$ then $\alpha \notin F(\beta,A)$ or $\beta \notin F(\alpha,A)$.

Notice that every element of $\mathcal{F}_\alpha$ is determined by a finite family of subsets of $\mathfrak{c}$, so, to get our space to be not collectionwise Hausdorff, we must consider all possible assignments $f : \alpha \mapsto \mathcal{A}_\alpha$ of finite families of subsets of $\mathfrak{c}$ and, somehow, ensure that there are disctinct α and β with $\alpha \in \bigcap_{A\in\mathcal{A}_\beta} F(\beta,A)$ and $\beta \in \bigcap_{A\in\mathcal{A}_\alpha} F(\alpha,A)$.

Our strategy for dealing with $2^{\mathfrak{c}}$ such assignments in only $\mathfrak{c}$ steps is based on the following idea. Take a countable elementary substructure M of the universe that has the assignment f in it and look at the restriction of f to M, i.e., the map $f^M : \mathfrak{c} \cap M \to [\mathcal{P}(M \cap \mathfrak{c})]^{<\omega}$ defined by $f^M(\beta) = \{A \cap M : A \in \mathcal{A}_\alpha\}$. There are only $\mathfrak{c}$ many such restrictions and they give us enough information to deal with all possible assignments using just $\mathfrak{c}$ many points.

So let $\{(A_\beta, f_\beta) : \beta \in P\}$ enumerate the family of all pairs of the form $(M \cap \mathfrak{c}, f^M)$, where M is an elementary substructure of $H(\theta)$ and where we assume that $P \subseteq \mathfrak{c}$ and the enumeration is such that always $A_\beta \subseteq \beta$.

Given (A_β, f_β) define a function $g_\beta : \mathbb{N} \to A_\beta$, as follows. Assume $g_\beta \restriction n_k$ is known and put $\mathcal{B}_k = \bigcup_{i<n_k} f_\beta(g_\beta(i))$, so $\mathcal{B}_k$ is a finite family of subsets of A_β. For every function $\chi : \mathcal{B}_k \to \{0,1\}$ choose, if possible, a point α_χ not in $\{g_\beta(i) : i < n_k\}$ such that for all $B \in \mathcal{B}_k$ we have $\alpha_\chi \in B$ iff $\chi(B) = 1$. Extend g_β to some $n_{k+1} > n_k$ so that $\{g_\beta(i) : n_k \leqslant i < n_{k+1}\}$ counts the set of α_χ's.

► **3**. The map g_β is one-to-one and defined on all of $\mathbb{N}$.

Define $f'_\beta : \mathbb{N} \to [\mathcal{P}(A)]^{<\omega}$ by $f'_\beta(i) = f_\beta(g_\beta(i)) \setminus \mathcal{B}_k$ for $n_k \leqslant i < n_{k+1}$. Now we can define the sets $J(\alpha, A)$:

$$J(\alpha, A) = \{\beta > \alpha : (\exists i)(\alpha = g_\beta(i) \wedge A \cap A_\beta \in f'_\beta(i))\}.$$

With this the definition of the $F(\alpha, A)$ is complete.

Let $f : \mathfrak{c} \to [\mathcal{P}(\mathfrak{c})]^{<\omega}$ be given and fix a countable elementary substructure M of the universe with $f \in M$. Fix β with $\mathfrak{c} \cap M = A_\beta$ and $f^M = f_\beta$.

► **4**. There is a k such that $A \in f(\beta)$ and $A \cap M \in f'_\beta(i)$ imply $i < n_k$.

Define $\chi : \mathcal{B}_k \to \{0,1\}$ by: if $i < n_k$ and $B \in f_\beta(g_\beta(i))$ and then $\chi(B \cap M) = 1$ iff $\beta \in B$.

► **5**. a. χ is well-defined, i.e., there are no $B, C \in \mathcal{B}_k$ with $B \cap M = C \cap M$ and $\beta \in B \setminus C$. *Hint*: elementarity.
b. α_χ is defined. *Hint*: elementarity.

► **6**. $\beta \in F(\alpha_\chi, A)$, whenever $A \in f(\alpha_\chi)$. Fix $j \in [n_k, n_{k+1})$ with $\alpha_\chi = g_\beta(j)$.
a. If $A \cap M \in f'_\beta(i)$ for some $i < n_k$ then $\beta \in A$ iff $\alpha_\chi \in A$, hence $\beta \in I(\alpha_\chi, A)$.
b. If $A \cap M \in f'_\beta(j)$ then $\beta \in J(\beta, A)$.

► **7**. $\alpha_\chi \in F(\beta, A)$, whenever $A \in f(\beta)$.
a. If $A \cap M \in f'_\beta(i)$ for some $i < n_k$ then $\beta \in A$ iff $\alpha_\chi \in A$, hence $\alpha_\chi \in I(\beta, A)$.
b. If $A \cap M \notin f'_\beta(i)$ for any $i < n_k$ then $A \cap M \notin f'_\beta(j)$ and so $\beta \notin J(\alpha_\chi, A)$, whence $\alpha_\chi \in F(\beta, A)$.

2. Balogh's example

Balogh's example is, to some extent, similar in spirit to Rudin's example but much more complicated.

The underlying set of our space X will be $\mathfrak{c}\times\omega$. As above we will construct, for each α, a filter $\mathcal{F}_\alpha$ and use these filters to define the topology: U is open iff whenever $(\alpha, n+1) \in U$ there is an $F \in \mathcal{F}_\alpha$ such that $\bigl\{(\beta, n) : \beta \in F\bigr\} \subseteq U$.

► **1**. a. For every n the set $U_n = \mathfrak{c} \times [0, n]$ is open.
b. For every n the set $L_n = \mathfrak{c} \times \{n\}$ is relatively discrete.

The hard part will be to ensure that the space is normal and not countably paracompact. Normality is handled much like in Rudin's example: there will be $F(\alpha, A)$ in $\mathcal{F}_\alpha$ such that $F(\alpha, A) \cap F(\beta, \mathfrak{c} \setminus A) = \varnothing$ whenever $\alpha \in A$ and $\beta \notin A$. Countable paracompactness follows because, for every n, a closed set contained in $\mathfrak{c} \times n$ must be 'small', in fact so small that whenever we choose closed sets $F_n \subseteq \mathfrak{c} \times n$ for every n, their union will not even cover $\mathfrak{c} \times \{0\}$.

The following combinatorial lemma lies at the basis of the construction.

2.1. Lemma. *There is a map $c \mapsto d_c$ from ${}^{\mathfrak{c}}2$ to itself such that whenever $f : \mathfrak{c} \to \omega$, $g : \mathfrak{c} \to [{}^{\mathfrak{c}}2]^{<\omega}$ and $h : \mathfrak{c} \to [\mathfrak{c}]^{<\omega}$ are given we can find $\alpha < \beta$ in $\mathfrak{c}$ with $f(\alpha) = f(\beta)$, if $c \in g(\alpha)$ then $c(\alpha) = d_c(\beta)$, and $\beta \notin h(\alpha)$.*

The construction

Given the lemma, the construction proceeds as follows. For $\alpha \in \mathfrak{c}$, $s \in [{}^{\mathfrak{c}}2]^{<\omega}$ and $a \in [\mathfrak{c}]^{<\omega}$ put

$$F(\alpha, s, a) = \bigl\{\beta \in \mathfrak{c} : (\forall c \in s)\bigl(d_c(\beta) = c(\alpha)\bigr)\bigr\} \setminus a.$$

Furthermore, for each α, let $\mathcal{F}_\alpha$ be the family of all sets of the form $F(\alpha, s, a)$.

► **2**. $F(\alpha, s_1, a_1) \cap F(\alpha, s_2, a_2) = F(\alpha, s_1 \cup s_2, a_1 \cup a_2)$.

It is very well possible that $F(\alpha, s, a) = \varnothing$ for some α, s and a; for example when $c(\alpha) = 1$ and d_c is constantly 0: in that case $F(\alpha, \{c\}, \varnothing) = \varnothing$. We will see however that this does not happen too often.

Normality

The space is even hereditarily normal.

Let H and K be separated subsets of X, i.e., $H \cap \operatorname{cl} K = \operatorname{cl} H \cap K = \varnothing$. We have to find disjoint open sets around H and K.

► **3**. It suffices to find, for each n, open sets V_n and W_n with $H \cap L_n \subseteq V_n$ and $\operatorname{cl} V_n \cap K = \varnothing$, as well as $K \cap L_n \subseteq W_n$ and $\operatorname{cl} W_n \cap H = \varnothing$.
Hint: Let $V = \bigcup_n (V_m \setminus \bigcup_{m \leqslant n} \operatorname{cl} W_n)$ and $W = \bigcup_n (W_m \setminus \bigcup_{m \leqslant n} \operatorname{cl} V_n)$.

► **4**. Let $A \subseteq \mathfrak{c}$ and $n \in \omega$. Then $A \times \{n\}$ and $(\mathfrak{c} \setminus A) \times \{n\}$ have disjoint open neighbourhoods.
a. The statement holds for $n = 0$. *Hint*: See Exercise 2.1.
b. If the statement holds for n then it holds for $n + 1$. *Hint*: Let c be the characteristic function of A and show that $F(\alpha, \{c\}, \varnothing)$ and $F(\beta, \{c\}, \varnothing)$ are disjoint whenever $\alpha \in A$ and $\beta \notin A$. Look at $A' \times \{n\}$, where $A' = \bigcup_{\alpha \in A} F(\alpha, \{c\}, \varnothing)$.

▶ **5**. If $m < n$ then $K \cap L_m$ and $H \cap L_n$ have disjoint open neighbourhoods.

a. There are disjoint open sets O_K and O_H in U_m such that $L_n \cap \operatorname{cl} K \subseteq O_K$ and $L_n \setminus \operatorname{cl} K \subseteq O_H$.

b. The set $O_H^* = O_H \cup (U_n \setminus (U_m \cup \operatorname{cl} K))$ is open and contains $H \cap L_n$.

c. O_H^* and O_K are as required.

▶ **6**. There are disjoint open sets V_n and O around $H \cap L_n$ and K respectively.

a. There are disjoint open sets V_n and O' around $H \cap L_n$ and $K \cap U_n$ respectively. *Hint*: Apply the previous two exercises.

b. The set $O = O' \cup (X \setminus (U_n \cup \operatorname{cl} H))$ is open and as required.

Countable paracompactness

We call a subset A of $\mathfrak{c}$ *separated* if we can find for each $\alpha \in A$ a set $F_\alpha \in \mathcal{F}_\alpha$ such that $\alpha \notin F_\beta$ and $\beta \notin F_\alpha$ whenever $\alpha \neq \beta$ in A. A set is *σ-separated* if it is the union of countably many separated sets.

▶ **7**. $\mathfrak{c}$ is not σ-separated. *Hint*: Apply Lemma 2.1.

▶ **8**. Let $n \in \omega$ and $A \subseteq \mathfrak{c}$. Then $A \setminus \varphi(A)$ is separated, where $\varphi(A) = \bigl\{\alpha : (\alpha, n+1) \in \operatorname{cl}\bigl(A \times \{n\}\bigr)\bigr\}$.

▶ **9**. If $n \in \omega$ and F_n is closed and a subset of U_n then $A_n = \{\alpha : (\alpha, 0) \in F_n\}$ is the union of $n+1$ many separated sets. *Hint*: $\varphi^{n+1}(A_n) = \varnothing$.

▶ **10**. X is not countably paracompact.

Proof of Lemma 2.1

The proof of Lemma 2.1 is much like that in Section 1: we try to deal with $2^{\mathfrak{c}}$ many possibilities by looking at their restrictions to countable elementary substructures of $H(\theta)$, where θ is sufficiently big, larger than $2^{2^{\mathfrak{c}}}$ will work.

However, we need an extra twist to the construction. Assume we have f, g and h as in the lemma. We take two countable elementary substructures M and N of $H(\theta)$, with $f, g, h \in M$ and $M \in N$. We define $A = \mathfrak{c} \cap N$ and $B = \{c \restriction A : c \in {}^{\mathfrak{c}}2 \cap M\}$.

▶ **11**. If $\alpha \in N$ and $\beta \notin A$ then $\beta \notin h(\alpha)$.

▶ **12**.a. For every n the preimage $f^{\leftarrow}(n)$ belongs to M.

b. If $\beta \notin A$ and $f(\beta) = n$ then $f^{\leftarrow}(n)$ is uncountable.

These two exercises show that it is quite easy to find $\alpha < \beta$ with $f(\alpha) = f(\beta)$ and $\beta \notin h(\alpha)$: simply take β outside A and $\alpha \in A$ with $f(\alpha) = f(\beta)$.

To get, given β, an α such that $d_c(\beta) = c(\alpha)$ for all $c \in g(\alpha)$ we have to do more work.

▶ **13**. If $c \in {}^{\mathfrak{c}}2 \cap N \setminus M$ then $c \restriction A \notin B$. *Hint*: If $c' \in {}^{\mathfrak{c}}2 \cap M$ then $c' \neq c$, use elementarity.

For $\alpha \in \mathfrak{c}$ define $e_\alpha : g(\alpha) \to 2$ by $e_\alpha(c) = c(\alpha)$.

▶ **14**. The function e_α depends only on the restriction of g to N, defined by $g^N(\alpha) = \{c \restriction N : c \in g(\alpha)\}$.

▶ **15**. Let $E = g(\beta) \cap M$ and $n = f(\beta)$, and put $e = e_\beta \restriction E$.

a. The set $H = \{\gamma : f(\gamma) = n,\ E \subseteq g(\gamma) \text{ and } e = e_\gamma \restriction E\}$ belongs to M and is cofinal in $\mathfrak{c}$.

b. If F is a finite subset of M and $\alpha \in \mathfrak{c} \cap M$ then there is a $\gamma \in H \cap M$ with $\gamma > \alpha$ and $F \cap g(\gamma) \setminus E = \varnothing$.

c. Choose, in M, a maximal subset K of H such that $g(\gamma) \cap g(\delta) = E$ whenever $\gamma \neq \delta$ in K. Then K is uncountable. *Hint*: If K is countable then $K \subseteq M$; consider $K \cup \{\beta\}$.

d. If $\alpha \in K$ and $c \in E$ then $c(\alpha) = c(\beta)$.

This gives us a clue as to how to define the value $d_c(\beta)$ for certain c: if there is an $\alpha \in K$ with $c \in g(\alpha)$ then $d_c(\beta) = c(\beta) = c(\alpha)$ if $c \in E$ and $d_c(\beta) = c(\alpha)$ if $c \notin E$. If there is no such α then $d_c(\beta)$ is not important, so we set $d_c(\beta) = 0$. However, this assumes that we know M and N, whereas we need to define the d_c knowing only $\mathfrak{c} \cap M$, $\mathfrak{c} \cap N$ and the restrictions of f, g and h.

To give the true definition we let $\{(a_\beta, A_\beta, B_\beta, f_\beta, g_\beta, h_\beta) : \beta \in P\}$ enumerate the set of structures of the form

$$(\mathfrak{c} \cap M, \mathfrak{c} \cap N, \{c \restriction (\mathfrak{c} \cap N) : c \in M\}, f \restriction N, g^N, h \restriction N),$$

where $M, N \prec H(\theta)$, $M \in N$ and $f, g, h \in M$. Also, g^N is defined on A by $g^N(\alpha) = \{c \restriction (\mathfrak{c} \cap N) : c \in g(\alpha)\}$. We assume P and the enumeration are chosen so that always $A_\beta \subseteq \beta$.

Fix $\beta \in P$ and consider the βth structure $(a_\beta, A_\beta, B_\beta, f_\beta, g_\beta, h_\beta)$.

Inspired by Exercise 2.15 we consider triples (n, E, e), where $n \in \mathbb{N}$, $E \in [a_\beta]^{<\aleph_0}$ and $e : E \to \{0, 1\}$. For each such triple put

$$H(n, E, e) = \{\gamma \in A_\beta : f(\gamma) = n, g(\gamma) \cap B_\beta = E \text{ and } e = e_\gamma \restriction E\}.$$

Here we define e_γ as above: $e_\gamma(c) = c(\gamma)$ for $c \in g_\beta(\gamma)$.

Still using Exercise 2.15 as our guideline we consider the set I_β of those (n, E, e) for which $H(n, E, e)$ has an infinite subset $K(n, E, e)$ such that $g_\beta(\gamma) \cap g_\beta(\delta) = E$ whenever $\gamma \neq \delta$ in $K(n, E, e)$.

▶ **16**. There are an infinite set J_β in A_β and a function $u_\beta : J_\beta \to [{}^{A_\beta}2]^{<\aleph_0}$ with disjoint values such that for every $(n, E, e) \in I_\beta$ there are infinitely many $\gamma \in J_\beta \cap K(n, E, e)$ with $u_\beta(\gamma) = g_\beta(\gamma) \setminus E$.

Now we define the d_c:

1. if $\beta \in P$ and $c \restriction A_\beta \in B_\beta$ then set $d_c(\beta) = c(\beta)$;
2. if $\beta \in P$ and $c \restriction A_\beta \notin B_\beta$ but $c \restriction A_\beta \in u_\beta(\alpha)$ for a (unique) $\alpha \in J_\beta$ then set then $d_c(\beta) = c(\alpha)$;

3. in all other cases set $d_c(\beta) = 0$.

   This definition works.

► **17**. Let f, g and h be given and take M and N with $f, g, h \in M$, $M \in N$ and $M, N \prec H(\theta)$. Fix β with $(a_\beta, A_\beta, B_\beta, f_\beta, g_\beta, h_\beta) = (\mathfrak{c} \cap M, \mathfrak{c} \cap N, \{c \restriction A : c \in M\}, f \restriction N, g^N, h \restriction N)$. Let $n = f(\beta)$, $E = g(\beta \cap M)$ and $e = e_\beta \restriction E$. If $\alpha \in J_\beta \cap K(n, E, e)$ then $f(\alpha) = f(\beta)$, $\beta \notin h(\alpha)$ and $d_c(\beta) = c(\alpha)$ for all $c \in g_(\alpha)$.