Chapter 4

Dowker spaces

Products of normal spaces need not be normal; the square of the Sorgenfrey line is the best known example of this phenomenon. Lots of effort has gone into investigating what normal spaces do have normal products. The simplest case has turned out to be one of the most interesting: when is $X \times [0,1]$ normal? The spaces whose product with the unit interval $I = [0,1]$ is normal were characterized by Dowker and normal spaces whose product with I is not normal are called *Dowker spaces*.

1. Normality in products

We exhibit two non-normal products.

We first consider the square of the Sorgenfrey line. Remember that a local base at a point a is given by $\{(b,a] : b < a\}$.

▶ **1**. The Sorgenfrey line is normal. *Hint*: Given F and G choose for every $a \in \mathbb{S}$ a point $x_a < a$ such that $(x_a, a] \cap F = \varnothing$ if $x \notin F$ and $(x_a, a] \cap G = \varnothing$ if $x \notin G$; now let $U = \bigcup_{a\in F}(x_a, a]$ and $V = \bigcup_{a\in G}(x_a, a]$.

▶ **2**. The Sorgenfrey plane $\mathbb{S}^2$ is not normal. Let $P = \{(p,-p) : p \in \mathbb{P}\}$ and $Q = \{(q,-q) : q \in \mathbb{Q}\}$, where $\mathbb{P}$ and $\mathbb{Q}$ are the sets of irrational and rational numbers respectively.

a. P and Q are closed in $\mathbb{S}^2$.

Let U be an open set around P and for $n \in \mathbb{N}$ put $P_n = \{p \in \mathbb{P} : (p - 2^{-n}, p] \times (-p - 2^{-n}, -p] \subseteq U\}$.

b. There is an n such that $\operatorname{int}\operatorname{cl} P_n \neq \varnothing$ in the usual topology of the real line.

c. If $q \in \mathbb{Q} \cap \operatorname{int}\operatorname{cl} P_n$ then $(q,-q) \in \operatorname{cl} U$.

The next example is slightly better because, as we shall see, it shows better how the ingredients in Dowker's characterization appear.

▶ **3**. Consider the ordinal spaces ω_1 and $\omega_1 + 1$.

a. ω_1 and $\omega_1 + 1$ are normal.

b. $\omega_1 \times \omega_1 + 1$ is not normal. *Hint*: Consider $F = \{(\alpha,\alpha) : \alpha \in \omega_1\}$ and $G = \{(\alpha,\omega_1) : \alpha < \omega_1\}$; apply the Pressing-Down Lemma to show that $G \cap \operatorname{cl} U \neq \varnothing$ whenever U is an open set around F.

2. Borsuk's theorem

One of the reasons for wanting to know when $X \times I$ is normal is the following theorem, due to Borsuk.

2.1. THEOREM (Borsuk's Homotopy Extension Theorem). *Let X be a space such that $X \times I$ is normal, let A be a closed subspace of X and let $f, g : A \to S^n$ be continuous and homotopic. If f admits a continuous extension to X then so does g and the extensions may be chosen homotopic, in fact by a homotopy that extends the given homotopy between f and g.*

Two maps $f, g : X \to Y$ are *homotopic* if there is a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all x. We call H a *homotopy* between f and g. Thus Borsuk's theorem asserts that homotopies between maps can be extended provided one of the maps can be extended. Note the codomain, this the n-sphere, i.e, the subspace $\{x : \|x\| = 1\}$ of $\mathbb{R}^{n+1}$. For other codomains the proof is quite easy, e.g., for I^n the proof below finishes after the first step.

► **1**. Prove Borsuk's Homotopy Extension Theorem. Let $h : A \times I \to S^n$ be a homotopy between f and g and let $F : X \to S^n$ be an extension of f. Let $B = (A \times I) \cup (X \times \{0\})$ and define $k : B \to S^n$ by $k(x, t) = h(x, t)$ if $t > 0$ and $k(x, 0) = F(x)$.

a. The map k can be extended to a neighbourhood U of B. *Hint*: Extend k to $K : X \times I \to D$, where D is the massive ball, and let $U = \bigl\{(x, t) : K(x, t) \neq 0\bigr\}$; compose $K \restriction U$ with the projection with 0 as its centre.

b. There is a neighbourhood V of A such that $V \times I \subseteq U$.

c. There is a continuous function $l : X \to I$ such that $l(x) = 1$ for $x \in A$ and $l(x) = 0$ for $x \notin U$.

d. The map $H : (x, t) \mapsto K(x, l(x) \cdot t)$ is the desired homotopy.

3. Countable paracompactness

The property that characterizes normality of $X \times [0, 1]$ is *countable paracompactness*. To define it we must first introduce the following notion.

3.1. DEFINITION. A collection $\mathcal{A}$ of sets in a space X is *locally finite* if every point of X has a neighbourhood that intersects only finitely many elements of $\mathcal{A}$.

► **1**. If $\mathcal{A}$ is locally finite then $\operatorname{cl} \bigcup \mathcal{A} = \bigcup \{\operatorname{cl} A : A \in \mathcal{A}\}$.

Given two covers $\mathcal{A}$ and $\mathcal{B}$ of a set we say that $\mathcal{A}$ is a refinement of $\mathcal{B}$ if for every $A \in \mathcal{A}$ there is a $B \in \mathcal{B}$ such that $A \subseteq B$.

3.2. Definition. A space is *paracompact* is every open cover has a locally finite open refinement. It is *countably paracompact* is every countable open cover has a locally finite open refinement.

To get a feeling for what locally finite open refinements can do we have the following.

► **2**. a. A paracompact Hausdorff space is regular. *Hint*: Given a closed set F and $x \in X \setminus F$ choose, for every $y \in F$, an open set U_y with $y \in U_y$ and $x \notin \operatorname{cl} U_y$. Consider a locally finite open refinement of $\{X \setminus F\} \cup \{U_y : y \in F\}$.
b. A paracompact regular space is normal.

► **3**. A space is countably paracompact iff every countable open cover has a *countable* locally finite open refinement. *Hint*: If $\mathcal{V}$ is some locally finite open refinement of $\mathcal{U}$, choose $U_V \in \mathcal{U}$ with $V \subseteq U_V$ for every $V \in \mathcal{V}$. Put $W_U = \bigcup\{V : U_V = U\}$; then $\{W_U : U \in \mathcal{U}\}$ is locally finite and of cardinality not more than $\mathcal{U}$.

► **4**. Let $\mathcal{U}$ be a locally finite open cover of the normal space X. There is an open cover $\{V_U : U \in \mathcal{U}\}$ of X such that $\operatorname{cl} V_U \subseteq U$ for all U. *Hint*: Well-order $\mathcal{U}$ by $\prec$ and define V_U by recursion on U: first put $F_U = X \setminus \bigl(\bigcup_{W \prec U} V_W \cup \bigcup_{W \succ U} W\bigr)$ and then choose V_U with $F_U \subseteq V_U$ and $\operatorname{cl} V_U \subseteq U$.

The following theorem gives more characterizations of countable paracompactness.

3.3. Theorem. *The following are equivalent for a space X.*

1. *X is countably paracompact;*
2. *if $\{U_n : n \in \omega\}$ is an increasing open cover of X then there is a sequence $\{F_n : n \in \omega\}$ of closed sets with $F_n \subseteq U_n$ for all n and $X = \bigcup_n \operatorname{int} F_n$; and*
3. *if $\{F_n : n \in \omega\}$ is a decreasing sequence of closed sets in X with empty intersection then there is a sequence $\{U_n : n \in \omega\}$ of open sets with $F_n \subseteq U_n$ for all n and $\bigcap_n \operatorname{cl} U_n = \varnothing$.*

► **5**. Prove Theorem 3.3.
a. Prove 1 implies 2. *Hint*: Apply Exercise 3.3 to get $\{V_n : n \in \omega\}$ and put $F_n = X \setminus \bigcup_{m>n} V_m$.
b. Prove 2 implies 1. *Hint*: Given $\{U_n : n \in \omega\}$ apply 2 to $\{\bigcup_{m \leqslant n} U_n : n \in \omega\}$ and put $V_n = U_n \setminus \bigcup_{m<n} F_m$.
c. Prove 2 and 3 are equivalent.

The following is the characterization of countable paracompactness that is used most often.

► **6**. A normal space X is countably paracompact iff whenever $\{F_n : n \in \omega\}$ is a decreasing sequence of closed sets in X with empty intersection there is a sequence $\{U_n : n \in \omega\}$ of open sets with $F_n \subseteq U_n$ for all n and $\bigcap_n U_n = \varnothing$.

The following theorem is the promised characterization of normality of $X \times [0, 1]$.

3.4. THEOREM. *The product $X\times[0,1]$ is normal iff X is normal and countably paracompact.*

The proof is in the following two exercises.

► **7**. Assume $X \times [0,1]$ is normal.

a. X is normal.

b. X is countably paracompact. *Hint*: Let $\{F_n : n \in \omega\}$ be a decreasing sequence of closed sets with empty intersection. Let $F = \bigcup_n \bigl(F_n \times [2^{-n}, 1]\bigr)$ and $G = X \times \{0\}$.

► **8**. Assume X is normal and countably paracompact. Let F and G be closed and disjoint in $X \times [0,1]$. Let $\mathcal{B}$ be a countable base for the topology of $[0,1]$, closed under finite unions. For $x \in X$ let $F_x = \{t \in [0,1] : (x,t) \in F\}$ and define G_x similarly.

a. F_x and G_x are closed and disjoint.

b. For every x there is a $B \in \mathcal{B}$ with $F_x \subseteq B$ and $\operatorname{cl} B \cap G_x = \varnothing$.

c. If $B \in \mathcal{B}$ then $U_B = \{x : F_x \subseteq B \text{ and } \operatorname{cl} B \cap G_x = \varnothing\}$ is open in X.

Take a locally finite open cover $\{V_B : B \in \mathcal{B}\}$ of X with $\operatorname{cl} V_B \subseteq U_B$ for all B and let $V = \bigcup_{B\in\mathcal{B}}(V_B \times B)$.

d. $F \subseteq V$ and $\operatorname{cl} V \cap G = \varnothing$. *Hint*: $\{V_B \times B : B \in \mathcal{B}\}$ is locally finite.