Chapter 2

Elementarity

This chapter introduces and studies elementary substructures 'of the universe'.

1. Definability

Our first task is to define what *definable* subsets of a set are. Intuitively these are sets determined by some formula, and this is how we shall work with them, but the formal definition is more algebraic in nature. The problem is that we cannot quite formalize a quantifier like "there exists a formula".

1.1. DEFINITION. For $n \in \omega$ and $i, j < n$ set

1. $\mathrm{Proj}(A, R, n) = \{s \in A^n : (\exists t \in R)(t \restriction n = s)\}$;
2. $\mathrm{Diag}_=(A, n, i, j) = \{s \in A^n : s(i) = s(j)\}$; and
3. $\mathrm{Diag}_\in(A, n, i, j) = \{s \in A^n : s(i) \in s(j)\}$.

Using these operations we define the *definable* relations on A, as follows. First by recursion on $k \in \omega$ and all n simultaneously define $\mathrm{Df}'(k, A, n)$ by

$$\begin{aligned}
\mathrm{Df}'(0, A, n) &= \{\mathrm{Diag}_=(A, n, i, j) : i, j < n\} \cup \{\mathrm{Diag}_\in(A, n, i, j) : i, j < n\} \\
\mathrm{Df}'(k + 1, A, n) &= \mathrm{Df}'(k, A, n) \cup \{A^n \setminus R : R \in \mathrm{Df}'(k, A, n)\} \\
&\qquad \cup \{R \cap S : R, S \in \mathrm{Df}'(k, A, n)\} \\
&\qquad \cup \{\mathrm{Proj}(A, R, n) : R \in \mathrm{Df}'(k, A, n + 1)\}
\end{aligned}$$

Once this is done we set $\mathrm{Df}(A, n) = \bigcup_{k\in\omega} \mathrm{Df}'(k, A, n)$. These are the *definable* n-ary relations on A.

The family of definable relations is closed under taking complements, intersections and projections.

1.2. LEMMA. *If* $R, S \in \mathrm{Df}(A, n)$ *then* $A^n \setminus R, R \cap S \in \mathrm{Df}(A, n)$ *and if* $R \in \mathrm{Df}(A, n + 1)$ *then* $\mathrm{Proj}(A, R, n) \in \mathrm{Df}(A, n)$.

Taking complements, intersections and projections, correspond to applying $\neg$, $\wedge$ and $\exists v_i$ to formulas. The following lemma makes this connection more explicit.

1.3. Lemma. *Let $\varphi(x_0, \ldots, x_{n-1})$ be a formula whose free variables are among $x_0, \ldots, x_{n-1}$. Then for every set A*

$$\{s \in A^n : \varphi^A\bigl(s(0), \ldots, s(n-1)\bigr)\} \in \mathrm{Df}(A, n).$$

In order to make sense of this lemma we must delve into the notion of a formula, explain what free variables are, and define $\varphi^A\bigl(s(0), \ldots, s(n-1)\bigr)$.

Formulas

We all have a good idea what a formula is and we usually know how to recognise one when we see it. However, when we want to treat formulas mathematically we have to formalise our 'good idea'. We begin by listing the basic symbols in the language of set theory. These are: $\wedge$, $\neg$, $\exists$, (,), $\in$, $=$ and infinitely many variables: v_i (one for each natural number i). Our formulas will be finite sequences of basic symbols.

1.4. Definition. The formulas of set theory are built up as follows:

1. for all natural numbers i and j the expressions $v_i \in v_j$ and $v_i = v_j$ are formulas; and
2. if φ and ψ are formulas then so are $(\varphi) \wedge (\psi)$, $\neg(\varphi)$ and $\exists v_i(\varphi)$ for any i.

Note the parentheses, these help to keep everything tidy. In practice we would consider $v_0 \in v_2 \wedge \neg(v_3 = v_4)$ to be a good formula (and we shall often do so) but when we want to prove something about formulas we shall replace it with its correct form $(v_0 \in v_2) \wedge \bigl(\neg(v_3 = v_4)\bigr)$. From elementary logic we know that the formulas allowed by Definition 1.4 express everything we want to express.

▶ **1**. The following are abbreviations for certain more complicated formulas: $\forall v_i(\varphi)$, $(\varphi) \vee (\psi)$, $(\varphi) \to (\psi)$, $(\varphi) \leftrightarrow (\psi)$, $v_i \notin v_j$ and $v_i \neq v_j$. Write down these complicated forms.

We shall explain most of the notions related to formulas by way of the following one

$$(1) \qquad \bigl(\exists v_0(v_0 \in v_1)\bigr) \wedge \bigl(\exists v_1(v_2 \in v_1)\bigr).$$

▶ **2**. A *subformula* of a formula φ is a consecutive sequence of symbols from φ that is itself a formula. Identify the subformulas of formula 1.

The *scope* of the occurrence of a quantifier $\exists v_i$ in a formula is the (unique) subformula beginning with that $\exists v_i$.

▶ **3**. a. Identify the scopes of $\exists v_0$ and $\exists v_1$ in formula 1.
b. Prove that the scope of an occurrence is well-defined.

An occurrence of a variable v_i in a formula is *bound* if it lies in the scope of an occurrence of $\exists v_i$ in that formula, otherwise it is *free*.

▶ **4**. Identify which occurrence of v_1 in formula 1 is free and which is bound.

The truth or falsity of a formula depends on its free-occurring variables, not on the bound variables. Therefore we would write formula 1 as $\varphi(v_1, v_2)$, to indicate that it is about the free variables v_1 and v_2. However, common usage is a bit more flexible: if necessary we will write our formula as, for example, $\varphi(v_0, v_1, v_2, v_3)$ to indicate that its free variables are *among* v_0, v_1, v_2 and v_3.

Now, if a, b, c and d are constants or other variables then $\varphi(a, b, c, d)$ is the result of replacing every *free* occurence of v_0, v_1, v_2 and v_3 by a, b, c and d respectively. Thus, $\varphi(4, 3, 2, 1)$ is

$$\bigl(\exists v_0(v_0 \in 3)\bigr) \wedge \bigl(\exists v_1(2 \in v_1)\bigr),$$

and $\varphi(4, v_0, v_5, 1)$ is

$$\bigl(\exists v_0(v_0 \in v_0)\bigr) \wedge \bigl(\exists v_1(v_5 \in v_1)\bigr).$$

The second substitution is unfortunate because it has changed the meaning of the first part of φ from "v_1 has an element" to "something is an element of itself". Such substitutions will not be allowed; we only consider *free* substitutions: a substitution $\varphi(y_1, y_2, y_3, y_4)$ is free if no free occurence of an original v_i is in the scope of a quantifier $\exists y_i$ (this only matters if y_i is a variable of course).

In Lemma 1.3 we substitute elements of A for the free occurences of the variables in φ^A; in that case there is no problem with bad substitutions: the elements of A are not variables.

Finally we define what φ^A (the *relativation* of φ to A) means:

1. $(v_i = v_j)^A$ is $v_i = v_j$ and $(v_i \in v_j)^A$ is $v_i \in v_j$;
2. $(\varphi \wedge \psi)^A$ is $\varphi^A \wedge \psi^A$ and $(\neg\varphi)^A$ is $\neg(\varphi^A)$; and
3. $\bigl(\exists v_i(\varphi)\bigr)^A$ is $\exists v_i\bigl((v_i \in A) \wedge (\varphi)^A\bigr)$.

Thus, informally, φ^A is φ with every $\exists v_i$ replaced by $\exists v_i \in A$.

▶ **5**. Give φ^A, where φ is formula 1.

Now we are ready to prove Lemma 1.3; we know what a formula is, we know what φ^A is and we know how substitutions work. We abbreviate the set $\bigl\{s \in A^n : \varphi^A\bigl(s(0), \ldots, s(n-1)\bigr)\bigr\}$ as $G(\varphi, A)$ and prove the lemma by induction on the length of φ.

▶ **6**. a. If φ is $x_i \in x_j$ then $G(\varphi, A) = \mathrm{Diag}_\in(A, n, i, j)$.
b. If φ is $x_i = x_j$ then $G(\varphi, A) = \mathrm{Diag}_=(A, n, i, j)$.
c. $G(\varphi \wedge \psi, A) = G(\varphi, A) \cap G(\varphi, A)$.
d. $G(\neg\varphi, A) = A^n \setminus G(\varphi, A)$.

If $\varphi = \exists y(\psi)$ then there are two cases: y is not one of the variables $x_0, \ldots, x_{n-1}$ or $y = x_j$ for some $j < n$.

e. If y is not one of $x_0, \ldots, x_{n-1}$ then $G(\varphi, A) = \mathrm{Proj}(A, G(\psi, A), n)$, where we write ψ as $\psi(x_0, \ldots, x_{n-1}, y)$.

In case, for example, $y = x_0$ take a variable z not occuring in φ, write the formula $\psi(z, x_1, \ldots, x_{n-1})$ as $\psi'(x_0, \ldots, x_{n-1}, z)$ and let φ' be $\exists z(\psi')$.

f. The substitution $x_0 \to z$ is free.

g. ψ and ψ' are logically equivalent, hence so are φ and φ'.
$G(\varphi, A) = G(\varphi', A) = \mathrm{Proj}(A, G(\psi', A), n)$.

2. Elementary substructures

In order to define what elementary substructures are we must count the definable relations.

2.1. Definition. By recursion on m, we define $\mathrm{En}(m, A, n)$, for all n simultaneously, as follows

1. If $m = 2^i \cdot 3^j$ and $i, j < n$ then $\mathrm{En}(m, A, n) = \mathrm{Diag}_\in(A, n, i, j)$.
2. If $m = 2^i \cdot 3^j \cdot 5$ and $i, j < n$ then $\mathrm{En}(m, A, n) = \mathrm{Diag}_=(A, n, i, j)$.
3. If $m = 2^i \cdot 3^j \cdot 5^2$ then $\mathrm{En}(m, A, n) = A^n \setminus \mathrm{En}(i, A, n)$.
4. If $m = 2^i \cdot 3^j \cdot 5^3$ then $\mathrm{En}(m, A, n) = \mathrm{En}(i, A, n) \cap \mathrm{En}(j, A, n)$.
5. If $m = 2^i \cdot 3^j \cdot 5^4$ then $\mathrm{En}(m, A, n) = \mathrm{Proj}(A, \mathrm{En}(i, A, n+1), n)$.
6. In all other cases $\mathrm{En}(m, A, n) = \varnothing$.

► **1**. For any A and n we have $\mathrm{Df}(A, n) = \{\mathrm{En}(m, A, n) : m \in \omega\}$.

a. $\forall n\bigl(\mathrm{En}(m, A, n) \in \mathrm{Df}(A, n)\bigr)$ for all m. *Hint*: by induction on m.

b. $\forall n\bigl(\mathrm{Df}'(k, A, n) \subseteq \{\mathrm{En}(m, A, n) : m \in \omega\}\bigr)$ for all k. *Hint*: by induction on k.

c. The set $\mathrm{Df}(A, n)$ is countable.

The proof of Lemma 1.3 yields the following improvement.

2.2. Lemma. *Let $\varphi(x_0, \ldots, x_{n-1})$ be a formula whose free variables are among $x_0, \ldots, x_{n-1}$. Then there is an m such that for every set A*

$$\bigl\{s \in A^n : \varphi^A\bigl(s(0), \ldots, s(n-1)\bigr)\bigr\} = \mathrm{En}(m, A, n).$$

Using the enumeration we define the relation $M \prec N$ between sets.

2.3. Definition. We say that M is an *elementary substructure* of N — notation $M \prec N$ — if $M \subseteq N$ and

$$\forall n, m\bigl(\mathrm{En}(m, M, n) = \mathrm{En}(m, N, n) \cap M^n\bigr).$$

The following Lemma connects this notion to formulas; Lemma 2.2 facilitates the proof.

2.4. Lemma. *Let $\varphi(x_0, \ldots, x_{n-1})$ be a formula whose free variables are among $x_0, \ldots, x_{n-1}$. Then $M \prec N$ implies*

$$\bigl\{s \in M^n : \varphi^M\bigl(s(0), \ldots, s(n-1)\bigr)\bigr\} = \bigl\{s \in N^n : \varphi^N\bigl(s(0), \ldots, s(n-1)\bigr)\bigr\} \cap M^n.$$

To get a feeling for what the definition and this lemma say we look at an important special case.

► **2**. If $a \in M \prec N$ and $a \cap N \neq \varnothing$ then $a \cap M \neq \varnothing$.
  a. Prove this from the definition. *Hint*: Note that the assumption says $a \in \mathrm{Proj}(N, \mathrm{Diag}_{\in}(N, 2, 1, 0), 1) \cap M$.
  b. Prove this using Lemma 2.4. *Hint*: Let $\varphi(v_0)$ be $\exists v_1(v_1 \in v_0)$ and note that the assumption says $a \in \{s \in N : \varphi^N(s)\} \cap M$.

A lot of arguments involving elementarity boil down to a clever application of this exercise: to see that something of the right kind is in M show that the set of things of the right kind belongs to M and that its intersection with N is nonempty.

Another special case is when $n = 0$. This is because $A^0 = \{\varnothing\}$ (only the empty function has domain 0). Therefore $\mathrm{En}(m, A, 0)$ is either 0 or 1. In Lemma 2.2 the case $n = 0$ corresponds to formulas without free variables, so-called *sentences*, for which φ^A is either false (if $\mathrm{En}(m, A, 0) = 0$) or true (if $\mathrm{En}(m, A, 0) = 1$). This leads to the following notion from Model Theory: A and B are *elementarily equivalent* if $\mathrm{En}(m, A, 0) = \mathrm{En}(m, B, 0)$ for all m; we write this as $A \equiv B$.

► **3**. If $A \prec B$ then $A \equiv B$.

► **4**. If m is of the form $2^i \cdot 3^j \cdot 5^4$ then $\mathrm{En}(m, A, 0) = 1$ iff $\mathrm{En}(i, A, 1) \neq \varnothing$. Therefore an elementary substructure of a nonempty set is nonempty.

► **5**. ω is the only elementary substructure of itself. Let $M \prec \omega$.
  a. $\varnothing \in M$. *Hint*: use the sentence $\exists x\bigl(\forall y(y \notin x)\bigr)$.
  b. If $n \in M$ then $n + 1 \in M$. *Hint*: use the formula $\exists y\bigl((x \in y) \land \forall z((x \in z) \to ((z = y) \lor (y \in z)))\bigr)$

The following fundamental result shows that a given structure has many elementary substructures. It is a special case of the Löwenheim-Skolem theorem from Model Theory.

2.5. Theorem. *Given N and $X \subseteq N$ there is an M such that $X \subseteq M$, $M \prec N$ and $|M| \leqslant \max(\aleph_0, |X|)$.*

► **6**. Prove Theorem 2.5. *Hint*: Let $\lhd$ be a well-ordering of N. For $m, n \in \omega$ define $H_{mn} : N^n \to N$ as follows. If m is of the form $2^i \cdot 3^j \cdot 5^4$ and $s \in \mathrm{En}(m, N, n) = \mathrm{Proj}(N, \mathrm{En}(i, N, n+1), n)$ then $H_{mn}(s)$ is the $\lhd$-first element of N such that $s^\frown x \in \mathrm{En}(i, N, n+1)$; in all other cases let $H_{mn}(s)$ be the $\lhd$-minimum of N. Let $X_0 = X$ and, recursively, let $X_{k+1} = X_k \cup \bigcup_{m,n\in\omega} H_{mn}[X_k^n]$. In the end let $M = \bigcup_{k\in\omega} X_k$.
  a. For all k we have $|X_k| \leqslant \max(\aleph_0, |X|)$ and also $|M| \leqslant \max(\aleph_0, |X|)$.
  b. For all m and n we have $\mathrm{En}(m, M, n) = \mathrm{En}(m, N, n) \cap M^n$. *Hint*: Induction on m, for all n simultaneously.

3. Elementary substructures of the Universe

In applications we work — intuitively — with elementary substructures of the set-theoretic universe but, because of things like 'the set of all sets', this can only be done on an intuitive level.

However, nothing prevents us from sharpening our intuition a bit before we put our method on a firm foundation. So, for the moment we treat V, the universe of all sets, as a set and fix an elementary substructure, M, of it. Observe that, because all sets are in V, for any formula φ the relativization φ^V is just φ itself.

What must be in M?

Certain things *must* be in M, simply because they are definable individuals. For instance, $\varnothing \in M$ because it is the unique set without elements. To see this note that the empty set is the only x that satisfies $\forall z(z \in x \longrightarrow z \neq z)$. Therefore if we write this formula as $\varphi(x)$ then the sentence $\exists x\varphi(x)$ is true in V, i.e., after applying Lemma 2.2 to get a number i for φ and setting $m = 2^i \cdot 3^0 \cdot 5^4$ we see that $\mathrm{En}(m, V, 0) = 1$ and hence $\mathrm{En}(m, M, 0) = 1$. Now apply Exercise 2.4 to see that both $\mathrm{En}(i, V, 1)$ and $\mathrm{En}(i, M, 1)$ are nonempty. But because $M \prec V$ we know that $\mathrm{En}(i, M, 1) = \mathrm{En}(i, V, 1) \cap M$. Now use uniqueness of $\varnothing$ to see that $\mathrm{En}(i, V, 1) = \{\varnothing\}$; therefore the only possibility is that $\mathrm{En}(i, M, 1) = \{\varnothing\}$ as well, and so $\varnothing \in M$.

▶ **1.** a. $\omega \subseteq M$. *Hint*: Apply Exercise 2.5 or do it now.
b. $\omega \in M$. *Hint*: ω is the (unique) minimal inductive set.
c. $\omega_1 \in M$. *Hint*: Find a formula that *defines* ω_1.

We can apply uniqueness to show that M is closed under various set-theoretic operations, the following exercise contains small sample.

▶ **2.** a. If $a \in M$ then $\bigcup a \in M$.
b. If $a, b \in M$ then $\{a, b\} \in M$ and so $a \cup b \in M$.
c. If $a \in M$ then $\mathcal{P}(a) \in M$.

The last part of this exercise gives rise to Skolem's paradox. In case M is countable the uncountable set $\mathcal{P}(\omega)$ belongs to M. Now, as M is an elementary substructure of the universe, all axioms of set theory are true in M, so M must somehow contain information that $\mathcal{P}(\omega)$ is uncountable. But $\mathcal{P}(\omega) \cap M$ is countable, so how can this be? The answer is that if f is a map from ω to $\mathcal{P}(\omega)$ that belongs to M then it is still subject to Cantor's diagonal argument, which yields a subset A of ω that is not in the range of f. So f belongs to $\{g : \exists x(x \in \mathcal{P}(\omega) \wedge x \notin \operatorname{ran} g)\} \cap M$, by elementarity f must therefore also belong to $\{g : \exists x(x \in M \wedge x \in \mathcal{P}(\omega) \wedge x \notin \operatorname{ran} g)\}$. This shows that no surjective map from ω onto $\mathcal{P}(\omega) \cap M$ can be a member of M.

► **3**. If F is a finite subset of M then $F \in M$. *Hint*: Fix $n \in \omega$ and a bijection $f : n \to F$. Apply Exercise 3.2 to show by induction that $\{f(j) : j < i\} \in M$ for every i.

► **4**. If $a \in M$ is countable then $a \subseteq M$. *Hint*: a belongs to $\bigl\{x : \exists f\bigl((\operatorname{dom} f = \omega) \wedge (\operatorname{ran} f = x)\bigr)\bigr\} \cap M$, so there is an $f \in M$ with $\operatorname{dom} f = \omega$ and $\operatorname{ran} f = x$. Use uniqueness to show that $f(i) \in M$ for all $i \in \omega$.

► **5**. If M is countable then $M \cap \omega_1$ is a countable ordinal.

Closed and unbounded sets and stationary sets

Countable M enable us to give fast proofs of facts about closed and unbounded sets and about stationary sets. As in the previous chapter we let $\mathcal{C}$ be the family of closed and unbounded subsets of ω_1. Also, for countable M we put $\delta_M = M \cap \omega_1$.

► **6**. a. $\mathcal{C} \in M$. *Hint*: Write down a formula that defines $\mathcal{C}$.
b. If M is countable and $C \in \mathcal{C} \cap M$ then $\delta_M \in C$. *Hint*: Every $\alpha < \delta_M$ belongs to M and to $\{\beta : \exists\gamma(\gamma \in C \wedge \gamma > \beta)\} \cap M$, deduce that $C \cap \delta_M$ is cofinal in δ_M.
c. If $\{C_n : n \in \omega\} \subseteq \mathcal{C}$ then $\bigcap_n C_n \in \mathcal{C}$. *Hint*: Let $\alpha \in \omega_1$ be arbitrary. Let M be countable with $\{C_n : n \in \omega\} \cup \alpha \subseteq M$. Consider δ_M.
d. If M is countable and $\langle C_\alpha : \alpha < \omega_1\rangle$ is a sequence in $\mathcal{C}$ that belongs to M then $\delta_M \in \triangle_\alpha C_\alpha$. *Hint*: If $\alpha \in M$ then $C_\alpha \in M$.

► **7**. Assume M is countable. If $S \in M \cap \mathcal{P}(\omega_1)$ and $\delta_M \in S$ then S is stationary. *Hint*: $S \in \bigl\{A \in M : \forall C\bigl((C \in M \wedge C \in \mathcal{C}) \to C \cap A \neq \varnothing\bigr)\bigr\}$.

4. Proofs using elementarity

We reprove some of the results from Chapter 1 using elementarity.

First the pressing-down lemma.

► **1**. Let $f : \omega_1 \to \omega_1$ be regressive and let M be countable with $f \in M$.
a. $\alpha = f(\delta_M) \in M$.
b. $S = \{\beta : f(\beta) = \alpha\}$ belongs to M and it is stationary.

Next the Δ-system lemma.

► **2**. Let $F = \langle F_\alpha : \alpha < \omega_1\rangle$ be a sequence of finite subsets of ω_1. Let M be countable with $F \in M$.
a. Let $R = F_{\delta_M} \cap \delta_M$, then $R \in M$.
b. The set $S = \{\alpha : R = F_\alpha \cap \alpha\}$ belongs to M.
c. The set $C = \{\alpha : (\forall\beta < \alpha)(\max F_\beta < \alpha)\}$ is closed and belongs to M; also $\delta_M \in C$, so C is unbounded as well.
d. The set $T = C \cap S$ is stationary and if $\alpha < \beta$ in T then $F_\alpha \cap F_\beta = R$.

And finally:

▶ **3**. Let $f : \omega_1 \to \mathbb{R}$ be continuous and let M be countable with $f \in M$.
a. Let $\varepsilon > 0$ and take $\alpha < \delta_M$ such that $\bigl|f(\beta) - f(\delta_M)\bigr| < \varepsilon$ whenever $\alpha < \beta \leqslant \delta_M$. Then $\bigl|f(\beta) - f(\gamma)\bigr| < 2\varepsilon$ whenever $\beta, \gamma > \alpha$.
b. f is constant on $[\delta_M, \omega_1)$.

5. Justification

Taking elementary substructures of the universe of all sets is not something that can be formalized in Set Theory. One can formalize the applications in Set Theory, however. To see this we must realize that the arguments use a limited supply of sets and simply take a large enough set that contains these sets and rework the argument inside that big set. The most popular of these large sets are called $H(\theta)$. We shall describe these and show how to work with them.

To define the $H(\theta)$ we must first define the *transitive closure* of sets. First, a set x is said to be *transitive* if it satisfies $(\forall y \in x)(y \subseteq x)$. Around every set we can find a smallest transitive set, as follows. Given x put $x_0 = x$ and, recursively, $x_{n+1} = x_n \cup \bigcup x_n$; in the end set $\operatorname{trcl} x = \bigcup_n x_n$.

▶ **1**. a. $\operatorname{trcl} x$ is transitive.
b. If y is transitive and $x \subseteq y$ then $\operatorname{trcl} x \subseteq y$.

Now we can define $H(\theta)$, for cardinal numbers θ:

$$H(\theta) = \{x : |\operatorname{trcl} x| < \theta\}.$$

Thus, e.g., $H(\aleph_1)$ is the set of all hereditarily countable sets.

▶ **2**. a. $\omega \in H(\aleph_1)$, $\omega_1 \in H(\aleph_2)$ and, generally, $\kappa \in H(\kappa^+)$ for all κ.
b. $\omega \subseteq H(\aleph_0)$, $\omega_1 \subseteq H(\aleph_1)$, and, generally, $\kappa \subseteq H(\kappa)$ for all κ.
c. $\mathcal{P}(\omega) \in H(\mathfrak{c}^+)$, $\mathcal{P}(\omega_1) \in H\bigl((2^{\aleph_1})^+\bigr)$, and, generally, $\mathcal{P}(\kappa) \in H\bigl((2^\kappa)^+\bigr)$ for all κ.
d. $\mathcal{P}(\omega) \subseteq H(\aleph_1)$, $\mathcal{P}(\omega_1) \subseteq H(\aleph_2)$, and, generally, $\mathcal{P}(\kappa) \subseteq H(\kappa^+)$ for all κ.

We check that the proofs from Section 4 can be done within relativelt small $H(\theta)$.

▶ **3**. a. Exercise 3.6 can be done for $M \prec H(\aleph_2)$. *Hint*: $\mathcal{C} \subseteq H(\aleph_2)$.
b. Exercise 3.7 can be done for $M \prec H(\aleph_2)$. *Hint*: Replace $C \in \mathcal{C}$ by a formula that expresses 'C is cub'.
c. The proofs of the pressing-down lemma and Δ-system lemma can be done with $M \prec H(\aleph_2)$.
d. Exercise 4.3 requires $M \prec H(\mathfrak{c}^+)$.

Theorem 2.5 admits refinements. The following will be needed in the proof of Arkhangel′skiĭ's theorem.

▶ **4**. Let $X \subseteq H(\theta)$ be of cardinality $\mathfrak{c}$ (or less). There is an M such that $X \subseteq M$, $M \prec H(\theta)$, $|M| \leqslant \mathfrak{c}$ and ${}^\omega M \subseteq M$. *Hint*: In the original proof redefine the X_k so that ${}^\omega X_k \subseteq X_{k+1}$.