Chapter 7

A Dowker space of size $\aleph_{\omega+1}$

In this chapter we show that Rudin's Dowker space has a closed subspace of cardinality $\aleph_{\omega+1}$ that is also a Dowker space.

1. A special sequence in $\prod_{n=1}^{\infty} \omega_n$

We shall not be working with the full product $\prod_{n=1}^{\infty} \omega_n$ but with a subproduct over a subset B of $\mathbb{N}$. For $B \subseteq \mathbb{N}$ we write P_B for $\prod_{n \in B}(\omega_n + 1)$ and $Q_B = \prod_{n \in B} \omega_n$. As before, for $x, y \in P_B$ we write

- $x < y$ if $(\forall n \in B)(x_n < y_n)$;
- $x \leqslant y$ if $(\forall n \in B)(x_n \leqslant y_n)$;
- $x <^* y$ if $\{n : x_n \geqslant y_n\}$ is finite;
- $x \leqslant^* y$ if $\{n : x_n > y_n\}$ is finite; and
- $x = *y$ if $\{n : x_n \neq y_n\}$ is finite.

For us the following result, which we quote without proof, will give us our Dowker subspace.

1.1. Theorem. *There are a subset B of $\mathbb{N}$ and a sequence $\langle x_\alpha : \alpha < \aleph_{\omega+1} \rangle$ in Q_B such that*

1. *the sequence is increasing with respect to $<^*$, i.e., $x_\alpha <^* x_\beta$ whenever $\alpha < \beta$; and*
2. *the sequence is cofinal, i.e., if $x \in Q_B$ then there is an α with $x <^* x_\alpha$.*

We call a sequence as in this theorem an $\aleph_{\omega+1}$-scale. We can improve the scale a bit.

▶ **1**. We can assume that for every δ of uncountable cofinality *if* $\{x_\alpha : \alpha < \delta\}$ has a least upper bound with respect to $\leqslant^*$ *then* x_δ is such a least upper bound.
Hint: Start with some $\aleph_{\omega+1}$-scale $\langle y_\alpha : \alpha < \aleph_{\omega+1} \rangle$. Given $\langle x_\alpha : \alpha < \delta \rangle$ let x_δ be a least upper bound in case $\operatorname{cf} \delta > \omega_0$ and a least upper bound exists, otherwise let $x_\delta = y_\beta$, where β is minimal with $x_\alpha <^* y_\beta$ for all α.

We shall need many least upper bounds. We write $B_k = B \cap (k, \omega)$ when $k \in \mathbb{N}$.

1.2. Lemma. *Let $k \geqslant m$ and let $\langle \alpha_\eta : \eta < \aleph_m \rangle$ be strictly increasing with $\delta = \sup_\eta \alpha_\eta < \aleph_{\omega+1}$. Let $\langle y_\eta : \eta < \aleph_m \rangle$ be a sequence in Q_{B_k} that is* *increasing with respect to* $<$ *and such that* $y_\eta =^* x_{\alpha_\eta}$ *for all* η, *and let* y *be the pointwise least upper bound of the* y_η.

1. $y \in Q_{B_k}$ *and it is a least upper bound of* $\langle x_{\alpha_\eta} : \eta < \aleph_m \rangle$;
2. $\operatorname{cf} y(n) = \aleph_m$ *for all* $n > k$; *and*
3. $y =^* x_\delta$.

▶ **2**. Prove Lemma 1.2.

a. $\operatorname{cf} y(n) = \aleph_m$ for $n > k$. *Hint*: The sequence is $<$-increasing.
b. $y \in Q_{B_k}$. *Hint*: Use $\operatorname{cf} y(n) = \aleph_m$.

Let y' be any upper bound of the x_{α_η} and $C = \{n \in B : n > k \text{ and } y'(n) < y(n)\}$.

c. For $n \in C$ there is η_n with $y_{\eta_n}(n) > y'(n)$.
d. $\sup\{\eta_n : n \in C\} = \eta < \aleph_m$.
e. $y_\eta(n) \geqslant y_{\eta_n}(n) > y'(n)$ for $n \in C$.
f. C is finite. *Hint*: $y_\eta =^* x_{\alpha_\eta} \leqslant^* y'$.
g. $y =^* x_\delta$.

2. The space

We use a subproduct of Rudin's space X, to wit the space X_B, where $X_B = \{x \restriction B : x \in X\}$.

▶ **1**. The space X_B is also a Dowker space.

The desired subspace Z of X_B is now easily defined:

$$Z = \{x \in X_B : (\exists \alpha < \aleph_{\omega+1})(x =^* x_\alpha)\}.$$

Some simple but useful observations about Z.

▶ **2**. The space Z has cardinality $\aleph_{\omega+1}$. *Hint*: If $\alpha < \aleph_{\omega+1}$ then $\{x \in X_B : x =^* x_\alpha\}$ has cardinality $\aleph_{\omega+1}$.

▶ **3**. If $x \in Z$ then there is a unique α with $x =^* x_\alpha$.

▶ **4**. If $x, y \in Z$ then $x <^* y$ or $y <^* x$ or $x =^* y$.

Z is collectionwise normal

To show that Z is (collectionwise) normal it suffices to show that Z is closed in X_B.

▶ **5**. Assume $m \leqslant k$ and let $\langle y_\eta : \eta < \aleph_m \rangle$ be a sequence in Z such that $\langle y_\eta \restriction B_k : \eta < \aleph_m \rangle$ is $<$-increasing and let y be its pointwise supremum. Then $y \in Z$. *Hint*: Apply Lemma 1.2; for every η there is one α_η with $y_\eta =^* x_{\alpha_\eta}$.

Now we show that Z is closed in X_B. Fix a point $t \in X_B$ that is in the closure of Z. For every $z \in Z$ put $E(z, t) = \{n \in B : z_n = t_n\}$. Of course we seek a z so that $E(z, t)$ is cofinite.

▶ **6**. If $z \in Z$ and $z \leqslant t$ then $E(z,t)$ is finite or cofinite. Define y by $y_n = 0$ if $n \in E(z,t)$ and $y_n = z_n$ if $n \notin E(z,t)$.
a. $y < t$ and there is $x \in Z \cap (y, t]$.
b. If $z <^* x$ then $E(z,t)$ is finite. *Hint*: $\{n : z_n < x_n\} \cap E(z,t) = \varnothing$.
c. If $x =^* z$ or $x <^* z$ then $E(z,t)$ is cofinite. *Hint*: $\{n : x_n < z_n\} \subseteq E(z,t)$.

For $w \subseteq B$ let $Z_w = \{z \in Z : z \leqslant t \text{ and } E(z,t) = w\}$. Note that Z_w is only nonempty if w is finite or cofinite.

▶ **7**. There are w such that $t \in \operatorname{cl} Z_w$. *Hint*: X_B is a P-space, show that $\operatorname{cl}\{z \in Z : z \leqslant t\} = \bigcup_w \operatorname{cl} Z_w$.

Fix some w such that $t \in \operatorname{cl} Z_w$; if this w is infinite we are done, so assume it is finite. Also put $M_m = \{n \in B : \operatorname{cf} t_n = \aleph_m\}$ and $M_{<m} = \bigcup_{i<m} M_i$.

▶ **8**. There is an m such that M_m is infinite.

Let m be minimal such that M_m is infinite. Let $k = \max(M_{<m} \cup w)$. For each $n \in M_n$ fix an increasing and cofinal sequence $\langle \gamma^n_\eta : \eta < \aleph_m \rangle$ in t_n.

▶ **9**. There is a sequence $\langle y_\eta : \eta < \aleph_m \rangle$ in Z_w such that
1. $y_\eta \leqslant t$ for all η;
2. if $\eta < \zeta$ then $y_\eta < y_\zeta < t$ on B_k; and
3. $y_\eta(n) \geqslant \gamma^n_\eta$ for $n \in B_k \cap M_m$.

▶ **10**. Let y be the pointwise supremum of the sequence $\langle y_\eta : \eta < \aleph_m \rangle$.
a. $y \in Z$.
b. $E(y,t)$ is cofinite. *Hint*: $M_n \subseteq E(y,t)$.
c. $t \in Z$.

Z is not countably paracompact

This is relatively easy. Let $F_n = \{z \in X_B : (\forall i \in B)(i \leqslant n \to z_i = \omega_i)\}$ for every n.

▶ **11**. If U is open in Z and $U \supseteq F_n \cap Z$ then there is $z \in Z$ with $(z,t] \cap Z \subseteq U$.
a. The set $V = U \cup (X_B \setminus Z)$ is open in X_B and $F_n \subseteq V$.
b. There is $x \in X_B$ with $(x,t] \cap X_B \subseteq V$.
c. There is $z \in Z$ with $z > x$.

▶ **12**. If U_n is open and $U_n \supseteq F_n$ for all n then $\bigcap_n U_n \neq \varnothing$.